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ORIGINAL PAPER

A family of ten-step methods with vanished phase-lag and its first derivative for the numerical solution of the Schrödinger equation

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Abstract A family of high algebraic order ten-step methods is obtained in this paper. The new developed methods have vanished phase-lag (the first one) and phase-lag and its first derivative (the second one). We apply the new developed methods to the resonance problem of the radial Schrödinger equation. The efficiency of the new proposed methods is shown via error analysis and numerical applications.

Keywords Numerical solution · Schrödinger equation · Multistep methods · Hybrid methods · Interval of periodicity · P-stability · Phase-lag · Phase-fitted · Derivatives of the phase-lag

1 Introduction

The radial Schrödinger equation can be written as:

$$y''(x) = [l(l+1)/x^2 + V(x) - k^2]y(x). \tag{1}$$

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It is known that many problems in theoretical physics and chemistry, material sciences, quantum mechanics and quantum chemistry, electronics etc. can be express via the above boundary value problem (see for example [1–4]).

We give the definitions of some terms of (1):

- The function $W(x) = l(l+1)/x^2 + V(x)$ is called *the effective potential*. This satisfies $W(x) \to 0$ as $x \to \infty$
- The quantity k^2 is a real number denoting *the energy*
- The quantity l is a given integer representing the *angular momentum*
- V is a given function which denotes the *potential*.

The boundary conditions are:

$$y(0) = 0 \tag{2}$$

and a second boundary condition, for large values of x, determined by physical considerations.

The last years an extended research on the development of numerical methods for the solution of the Schrödinger equation has been done. The aim of this study is the development of fast and reliable methods for the solution of the Schrödinger equation and related problems (see for example [5–87]. We mention the following bibliography:

- Phase-fitted methods and numerical methods with minimal phase-lag of Runge-Kutta and Runge-Kutta Nyström type have been developed in [10–27].
- In [28–33] exponentially and trigonometrically fitted Runge-Kutta and Runge-Kutta Nyström methods are obtained.
- Multistep phase-fitted methods and multistep methods with minimal phase-lag are developed in [38–61].
- Symplectic integrators are studied in [62–82].
- Exponentially and trigonometrically multistep methods have been developed in [83–110].
- Nonlinear methods have been studied in [111] and [112]
- Review papers have been written in [113–117]
- Special issues and Symposia in International Conferences have been created on this subject (see [118–124])

Generally the numerical methods for the approximate solution of the Schrödinger equation and related problems can be divided into two main categories:

1. Methods with constant coefficients
2. Methods with coefficients depending on the frequency of the problem [1].

The purpose of this paper is to use a new methodology for the development of numerical methods for the approximate solution periodic initial-value problems. The new

[1] When using a functional fitting algorithm for the solution of the radial Schrödinger equation, the fitted frequency is equal to: $\sqrt{|l(l+1)/x^2 + V(x) - k^2|}$.

methodology is based on the requirement of vanishing the phase-lag and its derivatives. Based on this new methodology we will develop two methods:

- The first one will have vanishing phase-lag (phase-fitted)
- The second one will have vanishing phase-lag and its first derivative

We will apply the new obtained methods in the numerical solution of the radial Schrödinger equation. The efficiency of the new methodology will be studied via the error analysis and the application to the specific potential.

More specifically, we will develop a family of implicit symmetric ten-step methods of twelfth algebraic order. The development of the new family of methods is based on the requirement of vanishing the phase-lag and its first derivative (see above). We will study the stability and the error of the methods of the two proposed new methods of the family. Finally, we will apply both of methods to the resonance problem. This is one of the most difficult problems arising from the radial Schrödinger equation. The paper is organized as follows.

- In Sect. 2 we present the theory of the new methodology.
- In Sect. 3 we present the development of the new family of methods.
- The error analysis is presented in Sect. 4.
- In Sect. 5 we will study the stability properties of the new obtained methods.
- In Sect. 6 the numerical results are presented.
- Finally, in Sect. 7 remarks and conclusions are discussed.

2 Phase-lag analysis of symmetric multistep methods

For the numerical solution of the initial value problem

$$p'' = f(x, p) \tag{3}$$

consider a multistep method with m steps which can be used over the equally spaced intervals $\{r_i\}_{i=0}^{m} \in [a, b]$ and $h = |r_{i+1} - r_i|$, $i = 0(1)m - 1$.

If the method is symmetric then $a_i = a_{m-i}$ and $b_i = b_{m-i}$, $i = 0(1)\lfloor \frac{m}{2} \rfloor$.

When a symmetric $2k$-step method, that is for $i = -k(1)k$, is applied to the scalar test equation

$$p'' = -\omega^2 p \tag{4}$$

a difference equation of the form

$$\begin{aligned} A_k(\mathrm{v})\, p_{n+k} &+ \cdots + A_1(\mathrm{v})\, p_{n+1} + A_0(\mathrm{v})\, p_n + \\ &+ A_1(\mathrm{v})\, p_{n-1} + \cdots + A_k(\mathrm{v})\, p_{n-k} = 0 \end{aligned} \tag{5}$$

is obtained, where $\mathrm{v} = \omega h$, h is the step length and $A_0(\mathrm{v}), A_1(\mathrm{v}), \ldots, A_k(\mathrm{v})$ are polynomials of v.

The characteristic equation associated with (5) is given by:

$$A_k(\mathrm{v})\,\lambda^k + \cdots + A_1(\mathrm{v})\,\lambda + A_0(\mathrm{v}) + A_1(\mathrm{v})\,\lambda^{-1} + \cdots + A_k(\mathrm{v})\,\lambda^{-k} = 0 \tag{6}$$

Theorem 1 [37] *The symmetric 2k-step method with characteristic equation given by (6) has phase-lag order r and phase-lag constant c given by*

$$-c\,\mathrm{v}^{r+2} + O(\mathrm{v}^{r+4}) = \frac{2\,A_k(\mathrm{v})\,\cos(k\,\mathrm{v}) + \cdots + 2\,A_j(\mathrm{v})\,\cos(j\,\mathrm{v}) + \cdots + A_0(\mathrm{v})}{2\,k^2\,A_k(\mathrm{v}) + \cdots + 2\,j^2\,A_j(\mathrm{v}) + \cdots + 2\,A_1(\mathrm{v})} \tag{7}$$

The formula proposed from the above theorem gives us a direct method to calculate the phase-lag of any symmetric $2k$- step method.

3 The new family of ten-step methods: construction of the new methods

We introduce the following family of methods to integrate $p'' = f(x, p)$:

$$\sum_{i=1}^{5} a_i\,\left(p_{n+i} + p_{n-i}\right) + a_0\,p_n = h^2\left[\sum_{i=1}^{5} b_i\,\left(p''_{n+i} + p''_{n-i}\right) + b_0\,p''_n\right] \tag{8}$$

where $a_5 = 1$.

3.1 First method of the family: a method with vanished phase-lag (phase-fitted)

Requiring the above method (8) to have the maximum algebraic order with one free parameter, the following relations are obtained:

$$\begin{aligned}
&a_0 = 0, \quad a_1 = 0, \quad a_2 = -1, \quad a_3 = 2, \quad a_4 = -2\\
&b_1 = \frac{2742857}{1555200} - \frac{5}{6}\,b_0, \quad b_2 = \frac{831701}{1360800} + \frac{10}{21}\,b_0, \quad b_3 = \frac{146717}{806400} - \frac{5}{28}\,b_0\\
&b_4 = \frac{3557441}{4082400} + \frac{5}{126}\,b_0, \quad b_5 = \frac{187585}{2612736} - \frac{1}{252}\,b_0
\end{aligned} \tag{9}$$

The application of the above method to the scalar test equation (4) gives the following difference equation:

$$\sum_{i=1}^{5} A_i(\mathrm{v})\,\left(p_{n+i} + p_{n-i}\right) + A_0(\mathrm{v})\,p_n = 0 \tag{10}$$

where $\mathrm{v} = \omega h$, h is the step length and $A_i(\mathrm{v}),\ i = 0(1)5$ are polynomials of v.

The characteristic equation associated with (10) is given by:

$$\sum_{i=1}^{5} A_i(\mathrm{v})\,\left(\lambda^i + \lambda^{-i}\right) + A_0(\mathrm{v}) = 0 \tag{11}$$

where

$$\begin{aligned}
A_0(\mathrm{v}) &= \mathrm{v}^2\, b_0 \\
A_1(\mathrm{v}) &= \mathrm{v}^2 \left(\frac{2742857}{1555200} - \frac{5}{6}\, b_0 \right) \\
A_2(\mathrm{v}) &= -1 + \mathrm{v}^2 \left(\frac{831701}{1360800} + \frac{10}{21}\, b_0 \right) \\
A_3(\mathrm{v}) &= 2 + \mathrm{v}^2 \left(\frac{146717}{806400} - \frac{5}{28}\, b_0 \right) \\
A_4(\mathrm{v}) &= -2 + \mathrm{v}^2 \left(\frac{3557441}{4082400} + \frac{5}{126}\, b_0 \right) \\
A_5(\mathrm{v}) &= 1 + \mathrm{v}^2 \left(\frac{187585}{2612736} - \frac{1}{252}\, b_0 \right)
\end{aligned} \tag{12}$$

By applying $k = 5$ in the formula (7), we have that the phase-lag is equal to:

$$phl = \frac{T_0}{T_1} \tag{13}$$

$$\begin{aligned}
T_0 = {} & 2 \left(1 + \mathrm{v}^2 \left(\frac{187585}{2612736} - \frac{1}{252}\, b_0 \right) \right) \cos(5\,\mathrm{v}) \\
& + 2 \left(-2 + \mathrm{v}^2 \left(\frac{3557441}{4082400} + \frac{5}{126}\, b_0 \right) \right) \cos(4\,\mathrm{v}) \\
& + 2 \left(2 + \mathrm{v}^2 \left(\frac{146717}{806400} - \frac{5}{28}\, b_0 \right) \right) \cos(3\,\mathrm{v}) \\
& + 2 \left(-1 + \mathrm{v}^2 \left(\frac{831701}{1360800} + \frac{10}{21}\, b_0 \right) \right) \cos(2\,\mathrm{v}) \\
& + 2\,\mathrm{v}^2 \left(\frac{2742857}{1555200} - \frac{5}{6}\, b_0 \right) \cos(\mathrm{v}) + \mathrm{v}^2 b_0 \\
T_1 = {} & 14 + 50\,\mathrm{v}^2 \left(\frac{187585}{2612736} - \frac{1}{252}\, b_0 \right) + 32\,\mathrm{v}^2 \left(\frac{3557441}{4082400} + \frac{5}{126}\, b_0 \right) \\
& + 18\,\mathrm{v}^2 \left(\frac{146717}{806400} - \frac{5}{28}\, b_0 \right) + 8\,\mathrm{v}^2 \left(\frac{831701}{1360800} + \frac{10}{21}\, b_0 \right) \\
& + 2\,\mathrm{v}^2 \left(\frac{2742857}{1555200} - \frac{5}{6}\, b_0 \right)
\end{aligned}$$

Demanding the phase-lag to be equal to zero we find out that:

$$b_0 = \frac{1}{259200}\, \frac{T_2}{T_3} \tag{14}$$

$$
\begin{aligned}
T_2 &= 4689625\ (\cos(\mathrm{v}))^5\,\mathrm{v}^2 + 65318400\ (\cos(\mathrm{v}))^5 \\
&\quad -65318400\ (\cos(\mathrm{v}))^4 + 28459528\ (\cos(\mathrm{v}))^4\,\mathrm{v}^2 \\
&\quad -48988800\ (\cos(\mathrm{v}))^3 - 2891012\ (\cos(\mathrm{v}))^3\,\mathrm{v}^2 \\
&\quad -23469322\ (\cos(\mathrm{v}))^2\,\mathrm{v}^2 + 57153600\ (\cos(\mathrm{v}))^2 \\
&\quad -4082400\cos(\mathrm{v}) + 6437243\,\mathrm{v}^2\cos(\mathrm{v}) \\
&\quad -4082400 + 1062338\,\mathrm{v}^2 \\
T_3 &= \mathrm{v}^2\Big((\cos(\mathrm{v}))^5 + 10\ (\cos(\mathrm{v}))^3 + 5\ \cos(\mathrm{v}) \\
&\quad -5\ (\cos(\mathrm{v}))^4 - 10\ (\cos(\mathrm{v}))^2 - 1\Big)
\end{aligned}
$$

For small values of $|\mathrm{v}|$ the formulae given by (14) are subject to heavy cancellations. In this case the following Taylor series expansions should be used:

$$
\begin{aligned}
b_0 &= \frac{18117277}{5702400} - \frac{547336457}{1482624000}\,\mathrm{v}^2 \\
&\quad + \frac{13099127}{8895744000}\,\mathrm{v}^4 + \frac{1122215903}{2016368640000}\,\mathrm{v}^6 \\
&\quad + \frac{411284674673}{14481559572480000}\,\mathrm{v}^8 + \frac{1674319402961}{1911565863567360000}\,\mathrm{v}^{10} \\
&\quad + \frac{54172151741}{4187239510671360000}\,\mathrm{v}^{12} - \frac{406647992425891}{872925240533778432000000}\,\mathrm{v}^{14} \\
&\quad - \frac{1769796744884513}{34567839525137625907200000}\,\mathrm{v}^{16} \\
&\quad - \frac{650688266276051}{227316858555326791680000000}\,\mathrm{v}^{18} + \cdots
\end{aligned} \tag{15}
$$

The behavior of the coefficients is given in the following Fig. 1.

The local truncation error of the new proposed method is given by:

$$
\mathrm{LTE} = -\frac{547336457\,h^{14}}{373621248000}\left(y_n^{(14)} + \omega^2\ y_n^{(12)}\right) \tag{16}
$$

3.2 Second method of the family: a method with vanished phase-lag and its first derivative

Now we require the method (8) to have the maximum algebraic order with two free parameters. So, the following relations are hold:

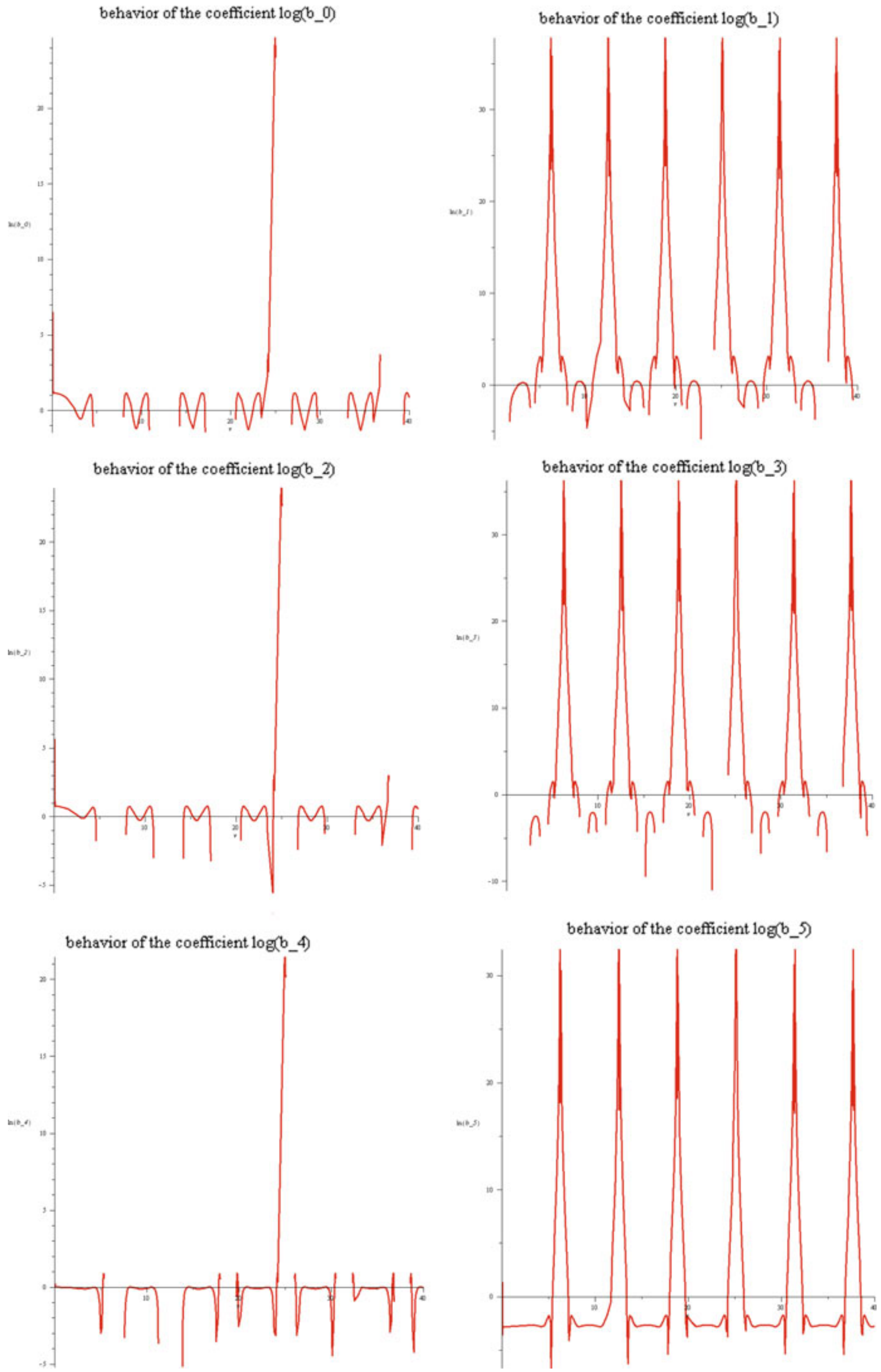

Fig. 1 Behavior of the coefficients of the new proposed method given by (9), (14) and (15) for several values of v

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = -1, \quad a_3 = 2, \quad a_4 = -2$$
$$b_2 = \frac{140387}{30240} - \frac{10}{7}\, b_0 - \frac{16}{7} b_1, \quad b_3 = -\frac{14423}{4480} + \frac{27}{14}\, b_1 + \frac{10}{7}\, b_0,$$
$$b_4 = \frac{200959}{90720} - \frac{16}{21}\, b_1 - \frac{25}{42}\, b_0, \quad b_5 = -\frac{50137}{362880} + \frac{5}{42}\, b_1 + \frac{2}{21}\, b_0 \qquad (17)$$

The application of the above method to the scalar test equation (4) gives the difference equation (10). The characteristic equation associated with (10) is given by (11) with:

$$A_0(\mathrm{v}) = \mathrm{v}^2\, b_0$$
$$A_1(\mathrm{v}) = \mathrm{v}^2\, b_1 \qquad (18)$$
$$A_2(\mathrm{v}) = -1 + \mathrm{v}^2 \left(\frac{140387}{30240} - \frac{10}{7}\, b_0 - \frac{16}{7}\, b_1\right)$$
$$A_3(\mathrm{v}) = 2 + \mathrm{v}^2 \left(-\frac{14423}{4480} + \frac{27}{14}\, b_1 + \frac{10}{7}\, b_0\right)$$
$$A_4(\mathrm{v}) = -2 + \mathrm{v}^2 \left(\frac{200959}{90720} - \frac{16}{21}\, b_1 - \frac{25}{42}\, b_0\right)$$
$$A_5(\mathrm{v}) = 1 + \mathrm{v}^2 \left(-\frac{50137}{362880} + \frac{5}{42}\, b_1 + \frac{2}{21}\, b_0\right) \qquad (19)$$

By applying $k = 5$ in the formula (7), we have that the phase-lag is equal to:

$$phl = \frac{T_4}{T_5} \qquad (20)$$
$$T_4 = \left(2\left(1 + \mathrm{v}^2\left(-\frac{50137}{362880} + \frac{5}{42}\, b_1 + \frac{2}{21}\, b_0\right)\right)\cos(5\,\mathrm{v})\right.$$
$$+2\left(-2 + \mathrm{v}^2\left(\frac{200959}{90720} - \frac{16}{21}\, b_1 - \frac{25}{42}\, b_0\right)\right)\cos(4\,\mathrm{v})$$
$$+2\left(2 + \mathrm{v}^2\left(-\frac{14423}{4480} + \frac{27}{14}\, b_1 + \frac{10}{7}\, b_0\right)\right)\cos(3\,\mathrm{v})$$
$$\left.+2\left(-1+\mathrm{v}^2\left(\frac{140387}{30240} - \frac{10}{7} b_0 - \frac{16}{7} b_1\right)\right)\cos(2\,\mathrm{v}) + 2\,\mathrm{v}^2 b_1 \cos(\mathrm{v}) + \mathrm{v}^2 b_0\right)$$
$$T_5 = \left(14 + 50\,\mathrm{v}^2\left(-\frac{50137}{362880} + \frac{5}{42}\, b_1 + \frac{2}{21}\, b_0\right)\right.$$
$$+32\,\mathrm{v}^2\left(\frac{200959}{90720} - \frac{16}{21}\, b_1 - \frac{25}{42}\, b_0\right)$$
$$+18\,\mathrm{v}^2\left(-\frac{14423}{4480} + \frac{27}{14}\, b_1 + \frac{10}{7}\, b_0\right)$$
$$\left.+8\,\mathrm{v}^2\left(\frac{140387}{30240} - \frac{10}{7}\, b_0 - \frac{16}{7}\, b_1\right) + 2\,\mathrm{v}^2 b_1\right)$$

The phase-lag's first derivative is given by:

$$\dot{phl} = \frac{T_6}{T_7} - \frac{T_8\, T_9}{T_7^2} \tag{21}$$

$$\begin{aligned}
T_6 = &\Bigg(4\,\mathrm{v}\left(-\frac{50137}{362880} + \frac{5}{42}\,b_1 + \frac{2}{21}\,b_0\right)\cos(5\,\mathrm{v}) \\
&-10\,\left(1 + \mathrm{v}^2\left(-\frac{50137}{362880} + \frac{5}{42}\,b_1 + \frac{2}{21}\,b_0\right)\right)\sin(5\,\mathrm{v}) \\
&+4\,\mathrm{v}\left(\frac{200959}{90720} - \frac{16}{21}\,b_1 - \frac{25}{42}\,b_0\right)\cos(4\,\mathrm{v}) \\
&-8\,\left(-2 + \mathrm{v}^2\left(\frac{200959}{90720} - \frac{16}{21}\,b_1 - \frac{25}{42}\,b_0\right)\right)\sin(4\,\mathrm{v}) \\
&+4\,\mathrm{v}\left(-\frac{14423}{4480} + \frac{27}{14}\,b_1 + \frac{10}{7}\,b_0\right)\cos(3\,\mathrm{v}) \\
&-6\,\left(2 + \mathrm{v}^2\left(-\frac{14423}{4480} + \frac{27}{14}\,b_1 + \frac{10}{7}\,b_0\right)\right)\sin(3\,\mathrm{v}) \\
&+4\,\mathrm{v}\left(\frac{140387}{30240} - \frac{10}{7}\,b_0 - \frac{16}{7}\,b_1\right)\cos(2\,\mathrm{v}) \\
&-4\,\left(-1 + \mathrm{v}^2\left(\frac{140387}{30240} - \frac{10}{7}\,b_0 - \frac{16}{7}\,b_1\right)\right)\sin(2\,\mathrm{v}) \\
&+4\,\mathrm{v}b_1\cos(\mathrm{v}) - 2\,\mathrm{v}^2 b_1\sin(\mathrm{v}) + 2\,\mathrm{v}b_0\Bigg) \\
T_7 = &\Bigg(14 + 50\,\mathrm{v}^2\left(-\frac{50137}{362880} + \frac{5}{42}\,b_1 + \frac{2}{21}\,b_0\right) \\
&+32\,\mathrm{v}^2\left(\frac{200959}{90720} - \frac{16}{21}\,b_1 - \frac{25}{42}\,b_0\right) \\
&+18\,\mathrm{v}^2\left(-\frac{14423}{4480} + \frac{27}{14}\,b_1 + \frac{10}{7}\,b_0\right) \\
&+8\,\mathrm{v}^2\left(\frac{140387}{30240} - \frac{10}{7}\,b_0 - \frac{16}{7}\,b_1\right) + 2\,\mathrm{v}^2 b_1\Bigg) \\
T_8 = &\Bigg(2\,\left(1 + \mathrm{v}^2\left(-\frac{50137}{362880} + \frac{5}{42}\,b_1 + \frac{2}{21}\,b_0\right)\right)\cos(5\,\mathrm{v}) \\
&+2\,\left(-2 + \mathrm{v}^2\left(\frac{200959}{90720} - \frac{16}{21}\,b_1 - \frac{25}{42}\,b_0\right)\right)\cos(4\,\mathrm{v}) \\
&+2\,\left(2 + \mathrm{v}^2\left(-\frac{14423}{4480} + \frac{27}{14}\,b_1 + \frac{10}{7}\,b_0\right)\right)\cos(3\,\mathrm{v}) \\
&+2\,\left(-1 + \mathrm{v}^2\left(\frac{140387}{30240} - \frac{10}{7}\,b_0 - \frac{16}{7}\,b_1\right)\right)\cos(2\,\mathrm{v}) \\
&+2\,\mathrm{v}^2 b_1\cos(\mathrm{v}) + \mathrm{v}^2 b_0\Bigg)
\end{aligned}$$

$$T_9 = \left(100\,\mathrm{v}\left(-\frac{50137}{362880} + \frac{5}{42}\,b_1 + \frac{2}{21}\,b_0\right)\right.$$
$$+64\,\mathrm{v}\left(\frac{200959}{90720} - \frac{16}{21}\,b_1 - \frac{25}{42}\,b_0\right)$$
$$+36\,\mathrm{v}\left(-\frac{14423}{4480} + \frac{27}{14}\,b_1 + \frac{10}{7}\,b_0\right)$$
$$\left.+16\,\mathrm{v}\left(\frac{140387}{30240} - \frac{10}{7}\,b_0 - \frac{16}{7}\,b_1\right) + 4\,\mathrm{v}b_1\right)$$

Demanding the phase-lag and its first derivative to be equal to zero we find out that:

$$b_0 = -\frac{1}{8640}\,\frac{T_{10}}{T_{11}} \tag{22}$$
$$T_{10} = 2419200\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^6 + 804932\,(\cos(\mathrm{v}))^6\,\mathrm{v}^3$$
$$+2661120\,(\cos(\mathrm{v}))^6\,\mathrm{v} + 2626322\,(\cos(\mathrm{v}))^5\,\mathrm{v}^3$$
$$-483840\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^5 + 2056320\,(\cos(\mathrm{v}))^5\,\mathrm{v}$$
$$-3749760\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^4 - 3840480\,(\cos(\mathrm{v}))^4\,\mathrm{v}$$
$$+2054125\,(\cos(\mathrm{v}))^4\,\mathrm{v}^3 - 548926\,(\cos(\mathrm{v}))^3\,\mathrm{v}^3$$
$$+665280\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^3 - 2963520\,(\cos(\mathrm{v}))^3\,\mathrm{v}$$
$$+1542240\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^2 - 1092584\,(\cos(\mathrm{v}))^2\,\mathrm{v}^3$$
$$+1406160\,\mathrm{v}\,(\cos(\mathrm{v}))^2 - 172276\,\mathrm{v}^3\cos(\mathrm{v})$$
$$-272160\,\cos(\mathrm{v})\sin(\mathrm{v}) + 907200\,\mathrm{v}\cos(\mathrm{v})$$
$$-120960\,\sin(\mathrm{v}) - 226800\,\mathrm{v} + 138647\,\mathrm{v}^3$$
$$T_{11} = \mathrm{v}^3\left((\cos(\mathrm{v}))^6 - 4\,(\cos(\mathrm{v}))^5 + 5\,(\cos(\mathrm{v}))^4\right.$$
$$\left.-5\,(\cos(\mathrm{v}))^2 + 4\,\cos(\mathrm{v}) - 1\right)$$
$$b_1 = \frac{1}{17280}\,\frac{T_{12}}{T_{11}} \tag{23}$$
$$T_{12} = 3870720\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^6 + 1307946\,(\cos(\mathrm{v}))^6\,\mathrm{v}^3$$
$$+4112640\,(\cos(\mathrm{v}))^6\,\mathrm{v} + 3870720\,(\cos(\mathrm{v}))^5\,\mathrm{v}$$
$$-483840\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^5 + 4121896\,(\cos(\mathrm{v}))^5\,\mathrm{v}^3$$
$$+3737637\,(\cos(\mathrm{v}))^4\,\mathrm{v}^3 - 6108480\,(\cos(\mathrm{v}))^4\,\mathrm{v}$$
$$-6289920\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^4 - 755314\,(\cos(\mathrm{v}))^3\,\mathrm{v}^3$$
$$-5503680\,(\cos(\mathrm{v}))^3\,\mathrm{v} + 846720\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^3$$
$$+2721600\,\sin(\mathrm{v})\,(\cos(\mathrm{v}))^2 - 2085490\,(\cos(\mathrm{v}))^2\,\mathrm{v}^3$$
$$+2404080\,\mathrm{v}\,(\cos(\mathrm{v}))^2 + 1632960\,\mathrm{v}\cos(\mathrm{v}) - 453600\,\cos(\mathrm{v})\sin(\mathrm{v})$$
$$-191382\,\mathrm{v}^3\cos(\mathrm{v}) + 215107\,\mathrm{v}^3 - 408240\,\mathrm{v} - 211680\,\sin(\mathrm{v})$$

For small values of |v| the formulae given by (22) and (23) are subject to heavy cancellations. In this case the following Taylor series expansions should be used:

$$
\begin{aligned}
b_0 &= \frac{18117277}{5702400} - \frac{547336457}{741312000}\,\mathrm{v}^2 \\
&\quad + \frac{813427897}{10674892800}\,\mathrm{v}^4 - \frac{38730780289}{27220976640000}\,\mathrm{v}^6 \\
&\quad - \frac{40679514679}{965437304832000}\,\mathrm{v}^8 - \frac{21098128575869}{1433674397675520000}\,\mathrm{v}^{10} \\
&\quad + \frac{4417854528839}{586213531493990400000}\,\mathrm{v}^{12} + \frac{20370634563077921}{480108882293578137600000}\,\mathrm{v}^{14} \\
&\quad + \frac{3962504447675167}{942759259776480706560000}\,\mathrm{v}^{16} \\
&\quad + \frac{41489033510961083}{113060227018307272704000000}\,\mathrm{v}^{18} + \cdots \\
b_1 &= -\frac{10081177}{11404800} + \frac{547336457}{889574400}\,\mathrm{v}^2 \\
&\quad - \frac{6413674837}{106748928000}\,\mathrm{v}^4 + \frac{970257427}{837568512000}\,\mathrm{v}^6 \\
&\quad + \frac{173246993783}{5111138672640000}\,\mathrm{v}^8 + \frac{195178446493}{16384850259148800}\,\mathrm{v}^{10} \\
&\quad - \frac{16628247827407}{2655184356689182720000}\,\mathrm{v}^{12} - \frac{917683407372509}{2602567375013492736000}\,\mathrm{v}^{14} \\
&\quad - \frac{52756649703962833}{1207407037150825755443200000}\,\mathrm{v}^{16} \\
&\quad - \frac{1764199674618507791}{859257725339135272550400000}\,\mathrm{v}^{18} + \cdots
\end{aligned}
\tag{24}
$$

The behavior of the coefficients is given in the following Fig. 2.

The local truncation error of the new proposed method is given by:

$$
\mathrm{LTE} = -\frac{547336457\,h^{14}}{373621248000}\left(y_n^{(14)} + 2\,\omega^2\,y_n^{(12)} + \omega^4\,y_n^{(10)}\right) \tag{25}
$$

4 Error analysis

We will study the following methods:

- The new proposed method of the family developed in Sect. 3.1 (mentioned as *PF*)
- The new proposed method of the family developed in Sect. 3.2 (mentioned as *PFDF*)
- The classical method of the family[2] (mentioned as *CL*)

[2] Classical is called the method of the family with constant coefficients.

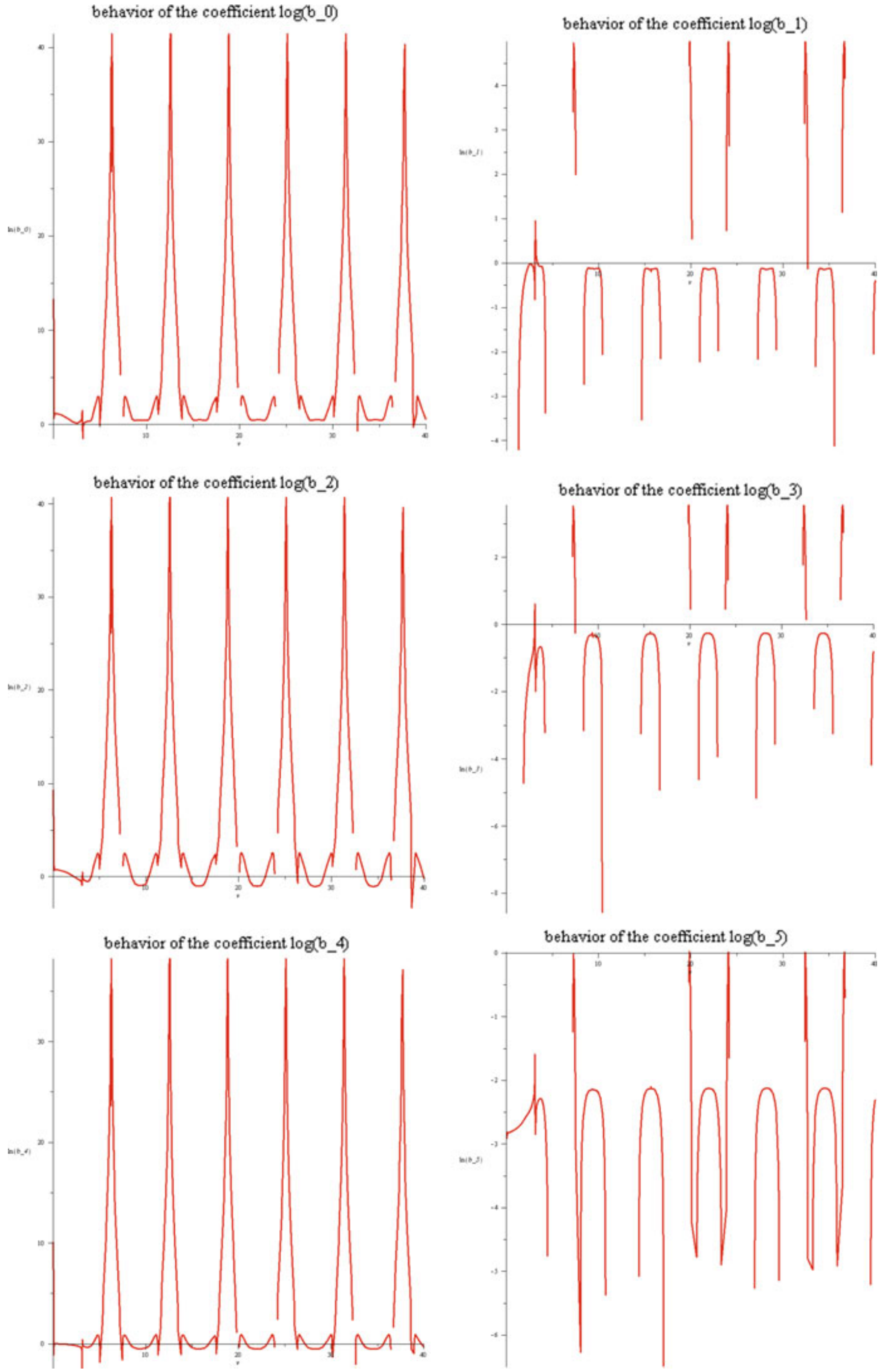

Fig. 2 Behavior of the coefficients of the new proposed method given by (17), (22), (23) and (24) for several values of v

The error analysis is based on the following steps:

– The radial time independent Schrödinger equation is of the form

$$q''(x) = f(x)\, q(x) \tag{26}$$

– Based on the paper of Ixaru and Rizea [116], the function $f(x)$ can be written in the form:

$$f(x) = g(x) + G \tag{27}$$

where $g(x) = V(x) - V_c = g$, where V_c is the constant approximation of the potential and $G = v^2 = V_c - E$.
– We express the derivatives $y_n^{(i)}$, $i = 2, 3, 4, \ldots,$ which are terms of the local truncation error formulae, in terms of the Eq. (26). The expressions are presented as polynomials of G.
– Finally, we substitute the expressions of the derivatives, produced in the previous step, into the local truncation error formulae.

Based on the procedure mentioned above and on the formulae:

$$\begin{aligned}
q_n^{(2)} &= (\mathrm{V}(x) - V_c + G)\; q(x)\\
q_n^{(4)} &= \left(\frac{d^2}{dx^2}\,\mathrm{V}(x)\right) q(x) + 2\left(\frac{d}{dx}\,\mathrm{V}(x)\right)\left(\frac{d}{dx}\,q(x)\right)\\
&\quad + (\mathrm{V}(x) - V_c + G)\left(\frac{d^2}{dx^2}\,q(x)\right)\\
q_n^{(6)} &= \left(\frac{d^4}{dx^4}\,\mathrm{V}(x)\right) q(x) + 4\left(\frac{d^3}{dx^3}\,\mathrm{V}(x)\right)\left(\frac{d}{dx}\,q(x)\right)\\
&\quad +3\left(\frac{d^2}{dx^2}\,\mathrm{V}(x)\right)\left(\frac{d^2}{dx^2}\,q(x)\right) + 4\left(\frac{d}{dx}\,\mathrm{V}(x)\right)^2 q(x)\\
&\quad +6\;(\mathrm{V}(x) - V_c + G)\left(\frac{d}{dx}\,q(x)\right)\left(\frac{d}{dx}\,\mathrm{V}(x)\right)\\
&\quad +4\;(\mathrm{U}(x) - V_c + G)\; q(x)\left(\frac{d^2}{dx^2}\,\mathrm{V}(x)\right)\\
&\quad + (\mathrm{V}(x) - V_c + G)^2\left(\frac{d^2}{dx^2}\,q(x)\right)\ldots
\end{aligned}$$

we obtain the expressions mentioned in Appendix.

We consider two cases in terms of the value of E:

– The Energy is close to the potential, i.e. $G = V_c - E \approx 0$. So only the free terms of the polynomials in G are considered. Thus for these values of G, the methods are of comparable accuracy. This is because the free terms of the polynomials in

G, are the same for the cases of the classical method and of the new developed methods.

– $G \gg 0$ or $G \ll 0$. Then $\mid G \mid$ is a large number. So, we have the following asymptotic expansions of the Eqs. (28) and (30).

4.1 The classical case of the family[3]

$$\mathrm{LTE}_{\mathrm{CL}} = h^{14} \left(-\frac{547336457}{373621248000}\, q\,(x)\;\, G^{7} + \cdots \right) \tag{28}$$

4.2 The new proposed method of the family developed in Sect. 3.1 (mentioned as *PF*)

$$\mathrm{LTE}_{\mathrm{PF}} = h^{14} \left(-\frac{547336457}{373621248000}\, g\,(x)\, q\,(x)\;\, G^{6} + \cdots \right) \tag{29}$$

4.3 The new proposed method of the family developed in Sect. 3.2 (mentioned as *PFDF*)

$$\begin{aligned} \mathrm{LTE}_{\mathrm{PFDF}} = h^{14} \Bigg[\Bigg(& -\frac{547336457}{17791488000} \left(\frac{d^2}{dx^2} g\,(x) \right) q\,(x) \\ & -\frac{547336457}{373621248000} \left(g\,(x) \right)^2 q\,(x) \\ & -\frac{547336457}{186810624000} \left(\frac{d}{dx} g\,(x) \right) \frac{d}{dx} q\,(x) \Bigg) G^{5} + \cdots \Bigg] \end{aligned} \tag{30}$$

From the above equations we have the following theorem:

Theorem 2 *For the Classical Method of the New Family of Methods the error increases as the seventh power of G. For the First Proposed Method of the New Family of Methods (with vanished phase-lag (phase-fitted)—developed in Sect. 3.1) the error increases as the sixth power of G. Finally, for the Second Proposed Method of the New Family of Methods (with vanished phase-lag and its first derivative—developed in Sect. 3.2) the error increases as the fifth power of G. So, for the numerical solution of the time independent radial Schrödinger equation the Second Proposed Method of the New Family of Methods (with vanished phase-lag and its first derivative) is the most accurate one, especially for large values of* $\mid G \mid = \mid V_c - E \mid$.

[3] Classical method of the family is the method of the family with constant coefficients which has the same algebraic order.

5 Stability analysis

We apply the new family of methods to the scalar test equation:

$$\phi'' = -t^2 \phi, \tag{31}$$

where $t \neq \omega$. We obtain the following difference equation:

$$\begin{aligned} & A_k(s)\,\phi_{n+k} + \cdots + A_1(s)\,\phi_{n+1} + A_0(s)\,\phi_n \\ & + A_1(s)\,\phi_{n-1} + \cdots + A_k(s)\,\phi_{n-k} = 0 \end{aligned} \tag{32}$$

where $s = t\,h$, h is the step length and $A_0(s), A_1(s), \ldots, A_k(s)$ are polynomials of s.

The characteristic equation associated with (32) is given by:

$$A_k(s)\,\theta^k + \cdots + A_1(s)\,\theta + A_0(s) + A_1(s)\,\theta^{-1} + \cdots + A_k(s)\,\theta^{-k} = 0 \tag{33}$$

Definition 1 (see [34]) A symmetric $2k$-step method with the characteristic equation given by (33) is said to have an *interval of periodicity* $\left(0, s_0^2\right)$ if, for all $s \in \left(0, s_0^2\right)$, the roots z_i, $i = 1, 2$ satisfy

$$z_{1,2} = e^{\pm i\,\zeta(t\,h)},\ |z_i| \leq 1,\ i = 3, 4 \tag{34}$$

where $\zeta(t\,h)$ is a real function of $t\,h$ and $s = t\,h$.

Definition 2 (see [34]) A method is called P-stable if its interval of periodicity is equal to $(0, \infty)$.

Definition 3 A method is called singularly almost P-stable if its interval of periodicity is equal to $(0, \infty) - S$[4] only when the frequency of the fitting is the same as the frequency of the scalar test equation, i.e. $\mathrm{v} = s$.

For the new proposed family of methods the difference equation (32) and the associated characteristic equation (33) are obtained for $k = 5$ with:

- $A_i(\mathrm{v})$, $i = 0(1)5$ given by (12) with b_0 obtained by (14) and (15) for the New Proposed Method of the Family Developed in Sect. 3.1 (mentioned as *PF*)
- $A_i(\mathrm{v})$, $i = 0(1)5$ given by (19) with b_0, b_1 obtained by (22), (23) and (24) for the New Produced Method of the Family Developed in Sect. 3.2 (mentioned as *PFDF*)

In Figs. 3 and 4 we present the $s - \mathrm{v}$ plane for the new methods produced in this paper. A shadowed area denotes the $s - \mathrm{v}$ region where the method is stable, while a white area denotes the region where the method is unstable.

In the case that the frequency of the scalar test equation is equal with the frequency of fitting, i.e. in the case that $\mathrm{v} = s$ (see the surroundings of the first diagonal of the $s - \mathrm{v}$ plane), we have that the interval of periodicity of the new obtained methods is equal to:

[4] Where S is a set of distinct points.

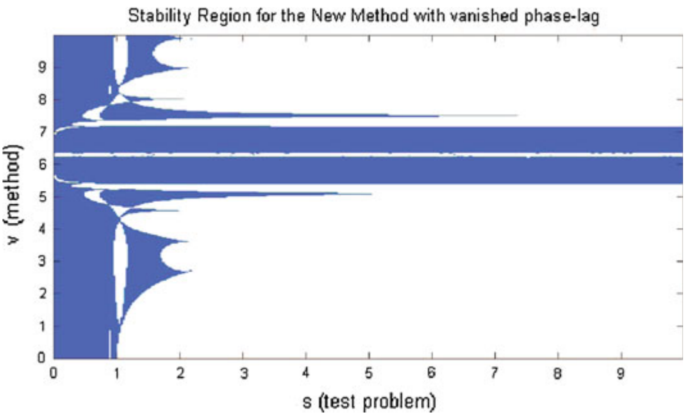

Fig. 3 $s - \mathrm{v}$ plane of the new proposed method of the Family Developed in Sect. 3.1 (mentioned as *PF*)

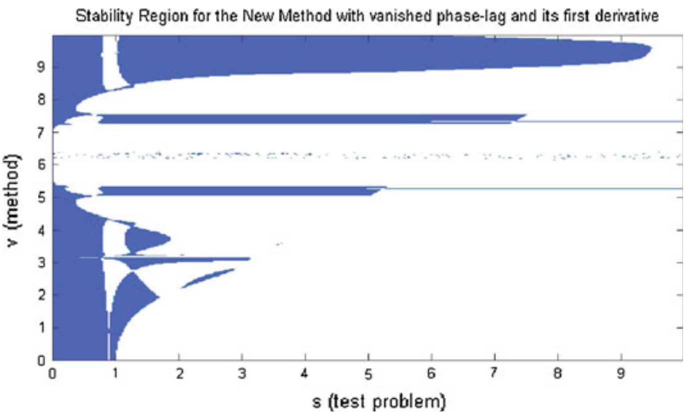

Fig. 4 $s - \mathrm{v}$ plane of the new proposed method of the Family Developed in Sect. 3.2 (mentioned as *PFDF*)

– (0, 0.8) for the Classical Method of the Family (mentioned as *CL*)
– (0, 1.2) for the New Proposed Method of the Family Developed in Sect. 3.1 (mentioned as *PF*)
– (0, 1.5) for the New Produced Method of the Family Developed in Sect. 3.2 (mentioned as *PFDF*)

Remark 1 For the solution of the Schrödinger equation the frequency of the exponential fitting is equal to the frequency of the scalar test equation. So, it is necessary to observe the surroundings of the first diagonal of the $s - \mathrm{v}$ plane.

From the above analysis we have the following theorem:

Theorem 3 *The method* (8) *with the coefficient* b_0 *obtained by* (14) *and* (15) *is of twelfth algebraic order, has the phase-lag equal to zero (phase-fitted) and has an interval of periodicity equals to:* (0, 1.2). *The method* (8) *with the coefficient* b_0, b_1

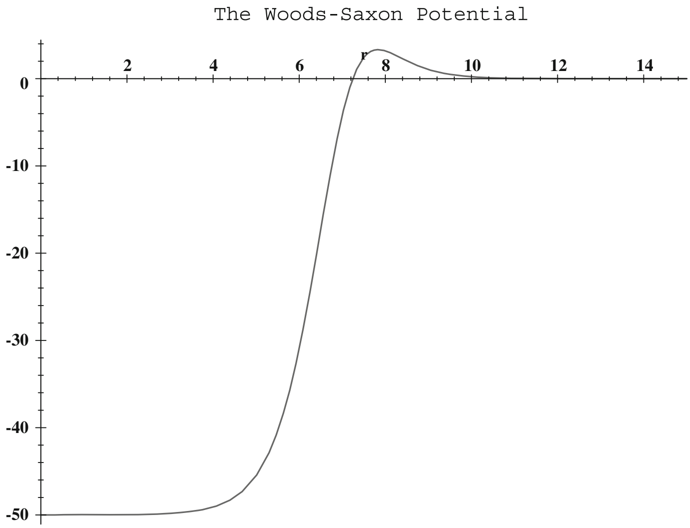

Fig. 5 The Woods-Saxon potential

obtained by (22), (23) *and* (24) *is of twelfth algebraic order, has the phase-lag and its first derivative equal to zero and has an interval of periodicity equals to:* (0, 1.5).

6 Numerical results

In this section we present the numerical of the application of the new developed methods to the resonance problem of the radial time independent Schrödinger equation.

Since our new methods are dependent from the parameter v, it is a requirement the determination of this parameter in order to be possible the application of the methods to the radial Schrödinger equation. For any problem of the radial Schrödinger equation given by (1) the parameter v is given by

$$\mathrm{v} = \sqrt{|q(x)|} = \sqrt{|V(x) - E|} \tag{35}$$

where $V(x)$ is the potential and E is the energy.

6.1 Woods-Saxon potential

For our numerical tests we use the well known Woods-Saxon potential given by

$$V(x) = \frac{u_0}{1+z} - \frac{u_0 z}{a\,(1+z)^2} \tag{36}$$

with $z = exp\left[(x - X_0)/a\right],\ u_0 = -50,\ a = 0.6$, and $X_0 = 7.0$.

The behavior of Woods-Saxon potential is shown in the Fig. 5.

It is well known that for some potentials, such as the Woods-Saxon potential, the definition of parameter v is not given as a function of x but it is based on some critical points which have been defined from the investigation of the appropriate potential (see for details [86]).

For the purpose of obtaining our numerical results it is appropriate to choose v as follows (see for details [86]):

$$v = \begin{cases} \sqrt{-50 + E}, & \text{for x} \in [0, 6.5 - 2\text{h}], \\ \sqrt{-37.5 + E}, & \text{for x} = 6.5 - h \\ \sqrt{-25 + E}, & \text{for x} = 6.5 \\ \sqrt{-12.5 + E}, & \text{for x} = 6.5 + h \\ \sqrt{E}, & \text{for x} \in [6.5 + 2\text{h}, 15] \end{cases} \tag{37}$$

6.2 Radial Schrödinger equation: the resonance problem

Consider the numerical solution of the radial time independent Schrödinger equation (1) in the well-known case of the Woods-Saxon potential (36). In order to solve this problem numerically we need to approximate the true (infinite) interval of integration by a finite interval. For the purpose of our numerical illustration we take the domain of integration as $x \in [0, 15]$. We consider Eq. (1) in a rather large domain of energies, i.e. $E \in [1, 1000]$.

In the case of positive energies, $E = k^2$, the potential dies away faster than the term $\frac{l(l+1)}{x^2}$ and the Schrödinger equation effectively reduces to

$$y''(x) + \left(k^2 - \frac{l(l+1)}{x^2}\right) y(x) = 0 \tag{38}$$

for x greater than some value X.

The above equation has linearly independent solutions $kxj_l(kx)$ and $kxn_l(kx)$ where $j_l(kx)$ and $n_l(kx)$ are the spherical Bessel and Neumann functions respectively. Thus the solution of Eq. (1) (when $x \to \infty$) has the asymptotic form

$$\begin{aligned} y(x) &\simeq Akxj_l(kx) - Bkxn_l(kx) \\ &\simeq AC \left[\sin\left(kx - \frac{l\pi}{2}\right) + \tan\delta_l \cos\left(kx - \frac{l\pi}{2}\right)\right] \end{aligned} \tag{39}$$

where δ_l is the phase shift, that is calculated from the formula

$$\tan\delta_l = \frac{y(x_2)S(x_1) - y(x_1)S(x_2)}{y(x_1)C(x_1) - y(x_2)C(x_2)} \tag{40}$$

for x_1 and x_2 distinct points in the asymptotic region (we choose x_1 as the right hand end point of the interval of integration and $x_2 = x_1 - h$) with $S(x) = kxj_l(kx)$ and $C(x) = -kxn_l(kx)$. Since the problem is treated as an initial-value problem, we need $y_0, y_i,\ i = 1(1)9$ before starting a ten-step method. From the initial condition we obtain y_0. The other values can be obtained using the Runge–Kutta–Nyström methods of Dormand et. al. (see [8]). With these starting values we evaluate at x_1 of the asymptotic region the phase shift δ_l.

For positive energies we have the so-called resonance problem. This problem consists either of finding the phase-shift δ_l or finding those E, for $E \in [1, 1000]$, at which $\delta_l = \frac{\pi}{2}$. We actually solve the latter problem, known as **the resonance problem** when the positive eigenenergies lie under the potential barrier.

The boundary conditions for this problem are:

$$y(0) = 0,\ y(x) = \cos\left(\sqrt{E}x\right) \text{ for large } x. \tag{41}$$

We compute the approximate positive eigenenergies of the Woods-Saxon resonance problem using:

- The Numerov's method which is indicated as **Method I**
- The Exponentially-fitted two-step method developed by Raptis and Allison [84] which is indicated as **Method II**
- The Exponentially-fitted two-step P-stable method developed by Kalogitatou and Simos [87] which is indicated as **Method III**
- The Exponentially-fitted four-step method developed by Raptis [85] which is indicated as **Method IV**
- The eight-step ninth algebraic order method developed by Quinlan and Tremaine [35] which is indicated as **Method V**
- The ten-step eleventh algebraic order method developed by Quinlan and Tremaine [35] which is indicated as **Method VI**
- The twelve-step thirteenth algebraic order method developed by Quinlan and Tremaine [35] which is indicated as **Method VII**
- The classical ten-step method of the family of methods mentioned in paragraph 3 which is indicated as **Method VIII**
- The new developed ten-step method with phase-lag equal to zero (phase-fitted) obtained in paragraph 3.1 which is indicated as **Method IX**.
- The new developed ten-step method with phase-lag and its first derivative equal to zero obtained in paragraph 3.2 which is indicated as **Method X**.

The computed eigenenergies are compared with exact ones. In Fig. 6 we present the maximum absolute error $\log_{10}(Err)$ where

$$Err = |E_{\text{calculated}} - E_{\text{accurate}}| \tag{42}$$

of the eigenenergy $E_2 = 341.495874$, for several values of NFE = Number of Function Evaluations. In Fig. 7 we present the maximum absolute error $\log_{10}(Err)$ where

$$Err = |E_{\text{calculated}} - E_{\text{accurate}}| \tag{43}$$

of the eigenenergy $E_3 = 989.701916$, for several values of NFE = Number of Function Evaluations.

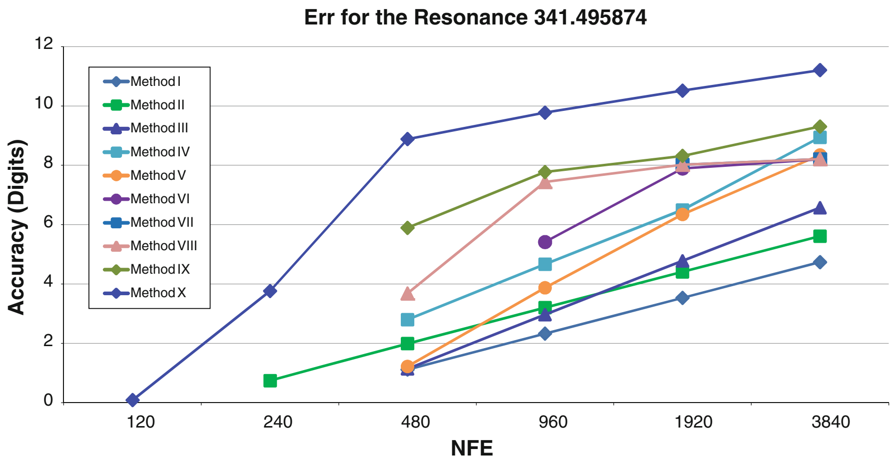

Fig. 6 Accuracy (Digits) for several values of NFE for the eigenvalue $E_2 = 341.495874$. The non-existence of a value of Accuracy (Digits) indicates that for this value of NFE, Accuracy (Digits) is less than 0

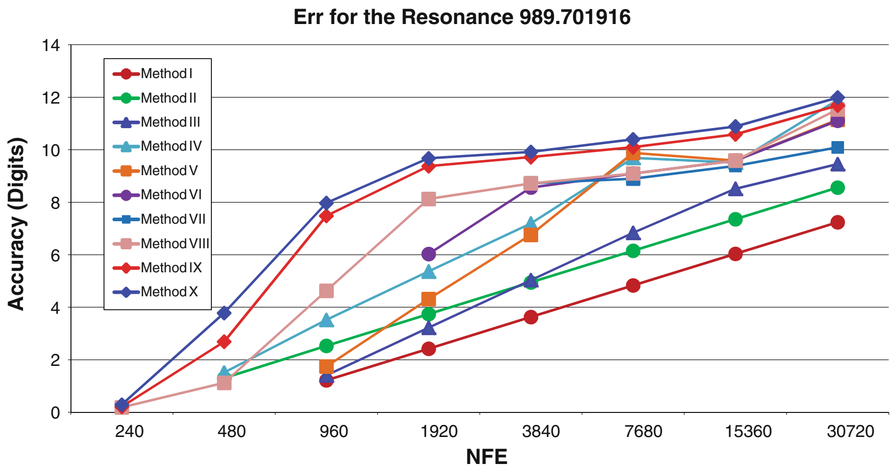

Fig. 7 Accuracy (Digits) for several values of NFE for the eigenvalue $E_3 = 989.701916$. The non-existence of a value of Accuracy (Digits) indicates that for this value of NFE, Accuracy (Digits) is less than 0

7 Conclusions

In the present paper we have developed an eight-step method of tenth algebraic order with phase-lag and its first derivative equal to zero.

We have applied the new method to the resonance problem of the one-dimensional Schrödinger equation.

Based on the results presented above we have the following conclusions:

- The Exponentially-fitted two-step method developed by Raptis and Allison [84] (denoted as Method II) is more efficient than the Numerov's Method (denoted Method I) and for low number of function evaluations is more efficient than the Exponentially-fitted two-step P-stable method developed by Kalogitatou and Simos [87] (denoted as Method III).
- The Exponentially-fitted two-step P-stable method developed by Kalogitatou and Simos [87] (denoted as Method III) is more efficient than the Exponentially-fitted two-step method developed by Raptis and Allison [84] (denoted as Method II) for high number of function evaluations.
- The Exponentially-fitted four-step method developed by Raptis [85] (denoted as Method IV) is more efficient than the Numerov' Method (denoted Method I), the Exponentially-fitted two-step method developed by Raptis and Allison [84] (denoted as Method II) and the Exponentially-fitted two-step P-stable method developed by Kalogitatou and Simos [87] (denoted as Method III)
- The eight-step ninth algebraic order method developed by Quinlan and Tremaine [35] (denoted as Method V) is more efficient than the Numerov' Method (denoted Method I), the Exponentially-fitted two-step method developed by Raptis and Allison [84] (denoted as Method II) and the Exponentially-fitted two-step P-stable method developed by Kalogitatou and Simos [87] (denoted as Method III) and less efficient than the Exponentially-fitted four-step method developed by Raptis [85] (denoted as Method IV)
- The ten-step eleventh algebraic order method developed by Quinlan and Tremaine [35] (denoted as Method VI) is more efficient than the Numerov' Method (denoted Method I), the Exponentially-fitted two-step method developed by Raptis and Allison [84] (denoted as Method II) and the Exponentially-fitted two-step P-stable method developed by Kalogitatou and Simos [87] (denoted as Method III) and the Exponentially-fitted four-step method developed by Raptis [85] (denoted as Method IV) for small number of function evaluations
- The twelve-step thirteenth algebraic order method developed by Quinlan and Tremaine [35] (denoted as Method VII) is more efficient than the Numerov' Method (denoted Method I), the Exponentially-fitted two-step method developed by Raptis and Allison [84] (denoted as Method II) and the Exponentially-fitted two-step P-stable method developed by Kalogitatou and Simos [87] (denoted as Method III), the Exponentially-fitted four-step method developed by Raptis [85] (denoted as Method IV) for small number of function evaluations, the eight-step ninth algebraic order method developed by Quinlan and Tremaine [35] (denoted as Method V) for small number of function evaluations and the ten-step eleventh algebraic order method developed by Quinlan and Tremaine [35] (denoted as Method VI) for small number of function evaluations
- The classical ten-step method of the family of methods mentioned in paragraph 3 (denoted as Method VIII) is more efficient than the Numerov' Method (denoted Method I), the Exponentially-fitted two-step method developed by Raptis and Allison [84] (denoted as Method II) and the Exponentially-fitted two-step P-stable method developed by Kalogitatou and Simos [87] (denoted as Method III). The method is also more efficient than the Exponentially-fitted four-step method developed by Raptis [85] (denoted as Method IV) for small number of function

evaluations, the eight-step ninth algebraic order method developed by Quinlan and Tremaine [35] (denoted as Method V) for small number of function evaluations, the ten-step eleventh algebraic order method developed by Quinlan and Tremaine [35] (denoted as Method VI) for small number of function evaluations and the twelve-step thirteenth algebraic order method developed by Quinlan and Tremaine [35] (denoted as Method VII) for small number of function evaluations

- The ten-step phase-fitted method of the family of methods developed in paragraph 3.1 (denoted as Method IX) is more efficient than the Numerov' Method (denoted Method I), the Exponentially-fitted two-step method developed by Raptis and Allison [84] (denoted as Method II) and the Exponentially-fitted two-step P-stable method developed by Kalogitatou and Simos [87] (denoted as Method III), the Exponentially-fitted four-step method developed by Raptis [85] (denoted as Method IV), the eight-step ninth algebraic order method developed by Quinlan and Tremaine [35] (denoted as Method V), the ten-step eleventh algebraic order method developed by Quinlan and Tremaine [35] (denoted as Method VI), the twelve-step thirteenth algebraic order method developed by Quinlan and Tremaine [35] (denoted as Method VII) and the classical ten-step method of the family of methods mentioned in paragraph 3 (denoted as Method VIII)
- Finally, the ten-step method of the family of methods with vanished phase-lag and its first derivative developed in paragraph 3.2 (denoted as Method X) is much more efficient than all the other methods.

All computations were carried out on a IBM PC-AT compatible 80486 using double precision arithmetic with 16 significant digits accuracy (IEEE standard).

Appendix

The classical case of the family

$$
\begin{aligned}
&\mathrm{LTE}_{\mathrm{CL}}\\
&= h^{14}\Bigg[-\frac{547336457}{373621248000}\,q(x)\,G^7-\frac{547336457}{53374464000}\,g(x)\,q(x)\,G^6\\
&\quad+\Bigg[-\frac{547336457}{8895744000}\left(\frac{d}{dx}g(x)\right)\frac{d}{dx}q(x)\\
&\quad-\frac{12588738511}{53374464000}\left(\frac{d^2}{dx^2}g(x)\right)q(x)-\frac{547336457}{17791488000}\,(g(x))^2\,q(x)\Bigg]G^5\\
&\quad+\Bigg[-\frac{547336457}{10674892800}\,(g(x))^3\,q(x)-\frac{547336457}{1779148800}\,g(x)\left(\frac{d}{dx}q(x)\right)\frac{d}{dx}g(x)\\
&\quad-\frac{547336457}{667180800}\left(\frac{d}{dx}g(x)\right)^2 q(x)-\frac{61849019641}{53374464000}\left(\frac{d^4}{dx^4}g(x)\right)q(x)\\
&\quad-\frac{12588738511}{10674892800}\,g(x)\,q(x)\,\frac{d^2}{dx^2}g(x)-\frac{547336457}{889574400}\left(\frac{d^3}{dx^3}g(x)\right)\frac{d}{dx}q(x)\Bigg]G^4
\end{aligned}
$$

$$+\left[-\frac{61849019641}{13343616000}\,g(x)\,q(x)\,\frac{d^4}{dx^4}g(x)\right.$$

$$-\frac{547336457}{889574400}\,(g(x))^2\left(\frac{d}{dx}q(x)\right)\frac{d}{dx}g(x)$$

$$-\frac{547336457}{222393600}\,g(x)\left(\frac{d}{dx}q(x)\right)\frac{d^3}{dx^3}g(x)-\frac{547336457}{166795200}\,g(x)\,q(x)\left(\frac{d}{dx}g(x)\right)^2$$

$$-\frac{351937341851}{53374464000}\left(\frac{d^2}{dx^2}g(x)\right)^2q(x)-\frac{636552299491}{373621248000}\left(\frac{d^6}{dx^6}g(x)\right)q(x)$$

$$-\frac{272026219129}{26687232000}\left(\frac{d}{dx}g(x)\right)q(x)\,\frac{d^3}{dx^3}g(x)$$

$$-\frac{43239580103}{26687232000}\left(\frac{d^5}{dx^5}g(x)\right)\frac{d}{dx}q(x)$$

$$-\frac{12588738511}{5337446400}\,(g(x))^2\,q(x)\,\frac{d^2}{dx^2}g(x)$$

$$-\frac{547336457}{111196800}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}q(x)\right)\frac{d^2}{dx^2}g(x)$$

$$\left.-\frac{547336457}{10674892800}\,(g(x))^4\,q(x)\right]G^3$$

$$+\left[-\frac{547336457}{111196800}\,(g(x))^2\,q(x)\left(\frac{d}{dx}g(x)\right)^2\right.$$

$$-\frac{344274631453}{13343616000}\left(\frac{d}{dx}g(x)\right)^2q(x)$$

$$\frac{d^2}{dx^2}g(x)-\frac{12588738511}{9340531200}\left(\frac{d^7}{dx^7}g(x)\right)\frac{d}{dx}q(x)$$

$$-\frac{191020423493}{13343616000}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}q(x)\right)$$

$$\frac{d^4}{dx^4}g(x)-\frac{272026219129}{8895744000}\,g(x)\,q(x)\left(\frac{d}{dx}g(x)\right)$$

$$\frac{d^3}{dx^3}g(x)-\frac{262174162903}{15567552000}\left(\frac{d^3}{dx^3}g(x)\right)^2$$

$$q(x)-\frac{469067343649}{16982784000}\left(\frac{d^2}{dx^2}g(x)\right)q(x)$$

$$\frac{d^4}{dx^4}g(x)-\frac{61849019641}{8895744000}\,(g(x))^2\,q(x)\,\frac{d^4}{dx^4}g(x)$$

$$-\frac{296109023237}{373621248000}\left(\frac{d^8}{dx^8}g(x)\right)q(x)$$

$$- \frac{547336457}{37065600} g(x) \left(\frac{d}{dx} q(x)\right) \left(\frac{d}{dx} g(x)\right)$$

$$\frac{d^2}{dx^2} g(x) - \frac{3831355199}{166795200} \left(\frac{d^2}{dx^2} g(x)\right) \left(\frac{d}{dx} q(x)\right)$$

$$\frac{d^3}{dx^3} g(x) - \frac{547336457}{148262400} (g(x))^2 \left(\frac{d}{dx} q(x)\right) \frac{d^3}{dx^3} g(x)$$

$$- \frac{636552299491}{124540416000} g(x) q(x) \frac{d^6}{dx^6} g(x)$$

$$- \frac{43239580103}{8895744000} g(x) \left(\frac{d}{dx} q(x)\right) \frac{d^5}{dx^5} g(x)$$

$$- \frac{547336457}{889574400} (g(x))^3 \left(\frac{d}{dx} q(x)\right) \frac{d}{dx} g(x)$$

$$- \frac{1395160628893}{93405312000} \left(\frac{d}{dx} g(x)\right) q(x) \frac{d^5}{dx^5} g(x)$$

$$- \frac{547336457}{148262400} \left(\frac{d}{dx} g(x)\right)^3 \frac{d}{dx} q(x) - \frac{351937341851}{17791488000} g(x) q(x) \left(\frac{d^2}{dx^2} g(x)\right)^2$$

$$- \frac{547336457}{17791488000} (g(x))^5 q(x) - \frac{12588738511}{5337446400} (g(x))^3 q(x) \frac{d^2}{dx^2} g(x)\Bigg] G^2$$

$$+ \Bigg[- \frac{6465685566541}{62270208000} \left(\frac{d}{dx} g(x)\right) q(x) \left(\frac{d^3}{dx^3} g(x)\right) \frac{d^2}{dx^2} g(x)$$

$$- \frac{547336457}{222393600} (g(x))^3 \left(\frac{d}{dx} q(x)\right) \frac{d^3}{dx^3} g(x)$$

$$- \frac{296109023237}{186810624000} g(x) q(x) \frac{d^8}{dx^8} g(x) - \frac{43239580103}{8895744000} (g(x))^2 \times \left(\frac{d}{dx} q(x)\right)$$

$$\frac{d^5}{dx^5} g(x) - \frac{547336457}{15966720} \left(\frac{d}{dx} g(x)\right)^2 q(x) \frac{d^4}{dx^4} g(x)$$

$$- \frac{1905278206817}{93405312000} \left(\frac{d^3}{dx^3} g(x)\right) q(x) \frac{d^5}{dx^5} g(x)$$

$$- \frac{1024066511047}{186810624000} \left(\frac{d}{dx} g(x)\right) q(x) \frac{d^7}{dx^7} g(x)$$

$$- \frac{394629585497}{8895744000} \left(\frac{d}{dx} g(x)\right) \left(\frac{d}{dx} q(x)\right) \left(\frac{d^2}{dx^2} g(x)\right)^2$$

$$- \frac{636552299491}{124540416000} (g(x))^2 q(x) \frac{d^6}{dx^6} g(x)$$

$$- \frac{272026219129}{8895744000} (g(x))^2 q(x) \left(\frac{d}{dx} g(x)\right) \frac{d^3}{dx^3} g(x)$$

$$- \frac{148328179847}{12454041600} \left(\frac{d^4}{dx^4} g(x)\right)^2 q(x) - \frac{12588738511}{37362124800} \left(\frac{d^9}{dx^9} g(x)\right) \frac{d}{dx} q(x)$$

$$- \frac{547336457}{102643200} \left(\frac{d}{dx} g(x)\right)^4 q(x) - \frac{43239580103}{1334361600} \left(\frac{d^3}{dx^3} g(x)\right) \times \left(\frac{d}{dx} q(x)\right)$$
$$\frac{d^4}{dx^4} g(x) - \frac{3831355199}{83397600} g(x) \left(\frac{d}{dx} q(x)\right) \left(\frac{d^2}{dx^2} g(x)\right) \frac{d^3}{dx^3} g(x)$$
$$- \frac{36671542619}{373621248000} \left(\frac{d^{10}}{dx^{10}} g(x)\right) q(x) - \frac{547336457}{166795200} (g(x))^3 q(x) \left(\frac{d}{dx} g(x)\right)^2$$
$$- \frac{547336457}{1779148800} (g(x))^4 \left(\frac{d}{dx} q(x)\right) \frac{d}{dx} g(x)$$
$$- \frac{547336457}{74131200} g(x) \left(\frac{d}{dx} q(x)\right) \left(\frac{d}{dx} g(x)\right)^3$$
$$- \frac{16967430167}{494208000} \left(\frac{d}{dx} g(x)\right)^2 \left(\frac{d}{dx} q(x)\right) \frac{d^3}{dx^3} g(x)$$
$$- \frac{191020423493}{6671808000} g(x) \left(\frac{d}{dx} q(x)\right) \left(\frac{d}{dx} g(x)\right) \frac{d^4}{dx^4} g(x)$$
$$- \frac{2663886536219}{124540416000} \left(\frac{d^2}{dx^2} g(x)\right)^3 q(x) - \frac{469067343649}{8491392000} g(x) q(x) \times \left(\frac{d^2}{dx^2} g(x)\right)$$
$$\frac{d^4}{dx^4} g(x) - \frac{1395160628893}{46702656000} g(x) q(x) \left(\frac{d}{dx} g(x)\right) \frac{d^5}{dx^5} g(x)$$
$$- \frac{344274631453}{6671808000} g(x) q(x) \left(\frac{d}{dx} g(x)\right)^2 \frac{d^2}{dx^2} g(x)$$
$$- \frac{200872479719}{8895744000} \left(\frac{d^2}{dx^2} g(x)\right) \left(\frac{d}{dx} q(x)\right) \frac{d^5}{dx^5} g(x)$$
$$- \frac{278594256613}{26687232000} \left(\frac{d}{dx} g(x)\right) \left(\frac{d}{dx} q(x)\right) \frac{d^6}{dx^6} g(x)$$
$$- \frac{61849019641}{13343616000} (g(x))^3 q(x) \frac{d^4}{dx^4} g(x)$$
$$- \frac{1575781659703}{124540416000} \left(\frac{d^2}{dx^2} g(x)\right) q(x)$$
$$\frac{d^6}{dx^6} g(x) - \frac{262174162903}{7783776000} g(x) q(x) \left(\frac{d^3}{dx^3} g(x)\right)^2$$
$$- \frac{12588738511}{4670265600} g(x) \left(\frac{d}{dx} q(x)\right) \frac{d^7}{dx^7} g(x)$$
$$- \frac{12588738511}{10674892800} (g(x))^4 q(x) \frac{d^2}{dx^2} g(x)$$
$$- \frac{547336457}{37065600} (g(x))^2 \left(\frac{d}{dx} q(x)\right) \left(\frac{d}{dx} g(x)\right) \frac{d^2}{dx^2} g(x)$$

$$-\frac{351937341851}{17791488000}\,(g(x))^2 q(x)\left(\frac{d^2}{dx^2}g(x)\right)^2$$

$$\left.-\frac{547336457}{53374464000}\,(g(x))^6 q(x)\right] G-\frac{547336457}{102643200}\,g(x)\,q(x)\left(\frac{d}{dx}g(x)\right)^4$$

$$-\frac{547336457}{8895744000}\,(g(x))^5\left(\frac{d}{dx}q(x)\right)\frac{d}{dx}g(x)$$

$$-\frac{636552299491}{373621248000}\,(g(x))^3 q(x)\,\frac{d^6}{dx^6}g(x)$$

$$-\frac{9304719769}{1213056000}\left(\frac{d^3}{dx^3}g(x)\right)\left(\frac{d}{dx}q(x)\right)\frac{d^6}{dx^6}g(x)$$

$$-\frac{547336457}{23328000}\left(\frac{d^2}{dx^2}g(x)\right)q(x)\left(\frac{d^3}{dx^3}g(x)\right)^2$$

$$-\frac{1067853427607}{31135104000}\left(\frac{d}{dx}g(x)\right)^2 q(x)$$

$$\left(\frac{d^2}{dx^2}g(x)\right)^2-\frac{547336457}{373621248000}\,(g(x))^7 q(x)$$

$$-\frac{250132760849}{6671808000}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}q(x)\right)\left(\frac{d^3}{dx^3}g(x)\right)^2$$

$$-\frac{547336457}{15966720}\,g(x)\,q(x)\left(\frac{d}{dx}g(x)\right)^2\frac{d^4}{dx^4}g(x)$$

$$-\frac{3831355199}{166795200}\,(g(x))^2\left(\frac{d}{dx}q(x)\right)\left(\frac{d^2}{dx^2}g(x)\right)\frac{d^3}{dx^3}g(x)$$

$$-\frac{278594256613}{26687232000}\,g(x)\left(\frac{d}{dx}q(x)\right)\left(\frac{d^6}{dx^6}g(x)\right)\frac{d}{dx}g(x)$$

$$-\frac{200872479719}{8895744000}\,g(x)\left(\frac{d}{dx}q(x)\right)\left(\frac{d^5}{dx^5}g(x)\right)\frac{d^2}{dx^2}g(x)$$

$$-\frac{43239580103}{1334361600}\,g(x)\left(\frac{d}{dx}q(x)\right)\left(\frac{d^4}{dx^4}g(x)\right)\frac{d^3}{dx^3}g(x)$$

$$-\frac{547336457}{471744000}\left(\frac{d^5}{dx^5}g(x)\right)^2 q(x)-\frac{12588738511}{741312000}\left(\frac{d}{dx}g(x)\right)^2\left(\frac{d}{dx}q(x)\right)$$

$$\frac{d^5}{dx^5}g(x)-\frac{1575781659703}{124540416000}\,g(x)\,q(x)\left(\frac{d^6}{dx^6}g(x)\right)\frac{d^2}{dx^2}g(x)$$

$$-\frac{547336457}{25272000}\left(\frac{d}{dx}g(x)\right)^3\left(\frac{d}{dx}q(x)\right)\frac{d^2}{dx^2}g(x)$$

$$-\frac{15872757253}{46702656000}\left(\frac{d}{dx}g(x)\right)q(x)\,\frac{d^9}{dx^9}g(x)-\frac{230428648397}{11321856000}\left(\frac{d^2}{dx^2}g(x)\right)^2$$

$$q(x)\frac{d^4}{dx^4}g(x) - \frac{9304719769}{11321856000}\left(\frac{d^2}{dx^2}g(x)\right)q(x)\frac{d^8}{dx^8}g(x)$$
$$-\frac{33387523877}{20756736000}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}q(x)\right)\frac{d^8}{dx^8}g(x)$$
$$-\frac{272026219129}{26687232000}(g(x))^3 q(x)\left(\frac{d}{dx}g(x)\right)\frac{d^3}{dx^3}g(x)$$
$$-\frac{547336457}{31135104000}\left(\frac{d^{11}}{dx^{11}}g(x)\right)\frac{d}{dx}q(x)$$
$$-\frac{3831355199}{80870400}\left(\frac{d^2}{dx^2}g(x)\right)^2\left(\frac{d}{dx}q(x)\right)\frac{d^3}{dx^3}g(x)$$
$$-\frac{219481919257}{31135104000}\left(\frac{d}{dx}g(x)\right)^2 q(x)\frac{d^6}{dx^6}g(x)$$
$$-\frac{12588738511}{37362124800}g(x)\left(\frac{d}{dx}q(x)\right)\frac{d^9}{dx^9}g(x)$$
$$-\frac{547336457}{667180800}(g(x))^4 q(x)\left(\frac{d}{dx}g(x)\right)^2$$
$$-\frac{148328179847}{12454041600}g(x)q(x)\left(\frac{d^4}{dx^4}g(x)\right)^2$$
$$-\frac{547336457}{28740096}\left(\frac{d}{dx}g(x)\right)^3 q(x)\frac{d^3}{dx^3}g(x)$$
$$-\frac{547336457}{148262400}(g(x))^2\left(\frac{d}{dx}q(x)\right)\left(\frac{d}{dx}g(x)\right)^3$$
$$-\frac{61849019641}{53374464000}(g(x))^4 q(x)\frac{d^4}{dx^4}g(x)$$
$$-\frac{1905278206817}{93405312000}g(x)q(x)\left(\frac{d^5}{dx^5}g(x)\right)\frac{d^3}{dx^3}g(x)$$
$$-\frac{344274631453}{13343616000}(g(x))^2 q(x)\left(\frac{d}{dx}g(x)\right)^2\frac{d^2}{dx^2}g(x)$$
$$-\frac{547336457}{111196800}(g(x))^3\left(\frac{d}{dx}q(x)\right)\left(\frac{d}{dx}g(x)\right)\frac{d^2}{dx^2}g(x)$$
$$-\frac{547336457}{889574400}(g(x))^4\left(\frac{d}{dx}q(x)\right)\frac{d^3}{dx^3}g(x) - \frac{296109023237}{373621248000}(g(x))^2 q(x)$$
$$\frac{d^8}{dx^8}g(x) - \frac{2663886536219}{124540416000}g(x)q(x)\left(\frac{d^2}{dx^2}g(x)\right)^3$$

$$-\frac{12588738511}{8491392000}\left(\frac{d^3}{dx^3}g(x)\right)q(x)\frac{d^7}{dx^7}g(x)$$
$$-\frac{394629585497}{8895744000}g(x)\left(\frac{d}{dx}q(x)\right)\left(\frac{d}{dx}g(x)\right)\left(\frac{d^2}{dx^2}g(x)\right)^2$$
$$-\frac{16967430167}{494208000}g(x)\left(\frac{d}{dx}q(x)\right)\left(\frac{d}{dx}g(x)\right)^2\frac{d^3}{dx^3}g(x)$$
$$-\frac{1024066511047}{186810624000}g(x)q(x)\left(\frac{d^7}{dx^7}g(x)\right)\frac{d}{dx}g(x)$$
$$-\frac{262174162903}{15567552000}(g(x))^2q(x)\left(\frac{d^3}{dx^3}g(x)\right)^2$$
$$-\frac{12588738511}{53374464000}(g(x))^5q(x)\frac{d^2}{dx^2}g(x)-\frac{351937341851}{53374464000}(g(x))^3q(x)$$
$$\left(\frac{d^2}{dx^2}g(x)\right)^2-\frac{12588738511}{9340531200}(g(x))^2\left(\frac{d}{dx}q(x)\right)\frac{d^7}{dx^7}g(x)$$
$$-\frac{43239580103}{26687232000}(g(x))^3\left(\frac{d}{dx}q(x)\right)\frac{d^5}{dx^5}g(x)$$
$$-\frac{23535467651}{11321856000}\left(\frac{d^4}{dx^4}g(x)\right)q(x)\frac{d^6}{dx^6}g(x)$$
$$-\frac{6020701027}{1415232000}\left(\frac{d^2}{dx^2}g(x)\right)\left(\frac{d}{dx}q(x)\right)\frac{d^7}{dx^7}g(x)$$
$$-\frac{449363231197}{12454041600}\left(\frac{d}{dx}g(x)\right)q(x)\left(\frac{d^4}{dx^4}g(x)\right)\frac{d^3}{dx^3}g(x)$$
$$-\frac{36671542619}{373621248000}g(x)q(x)\frac{d^{10}}{dx^{10}}g(x)$$
$$-\frac{547336457}{53913600}\left(\frac{d^4}{dx^4}g(x)\right)\left(\frac{d}{dx}q(x)\right)\frac{d^5}{dx^5}g(x)$$
$$-\frac{3831355199}{61776000}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}q(x)\right)\left(\frac{d^4}{dx^4}g(x)\right)\frac{d^2}{dx^2}g(x)$$
$$-\frac{1395160628893}{93405312000}(g(x))^2q(x)\left(\frac{d}{dx}g(x)\right)\frac{d^5}{dx^5}g(x)$$
$$-\frac{191020423493}{13343616000}(g(x))^2\left(\frac{d}{dx}q(x)\right)\left(\frac{d}{dx}g(x)\right)\frac{d^4}{dx^4}g(x)$$
$$-\frac{469067343649}{16982784000}(g(x))^2q(x)\left(\frac{d^2}{dx^2}g(x)\right)\frac{d^4}{dx^4}g(x)$$
$$-\frac{6465685566541}{62270208000}g(x)q(x)\left(\frac{d}{dx}g(x)\right)\left(\frac{d^2}{dx^2}g(x)\right)\frac{d^3}{dx^3}g(x)$$

$$-\frac{25724813479}{972972000}\left(\frac{d}{dx}g(x)\right)q(x)\left(\frac{d^5}{dx^5}g(x)\right)\frac{d^2}{dx^2}g(x)$$
$$-\frac{547336457}{373621248000}\left(\frac{d^{12}}{dx^{12}}g(x)\right)q(x)\Bigg] \quad (44)$$

The new proposed method of the family developed in Sect. 3.1 (mentioned as *PF*)

$$\begin{aligned}
&\mathrm{LTE}_{\mathrm{PF}}\\
&= h^{14}\Bigg[-\frac{547336457}{373621248000}g(x)q(x)\,G^6\\
&+\Bigg[-\frac{547336457}{62270208000}(g(x))^2q(x)-\frac{547336457}{31135104000}\left(\frac{d}{dx}g(x)\right)\frac{d}{dx}q(x)\\
&-\frac{547336457}{5660928000}\left(\frac{d^2}{dx^2}g(x)\right)q(x)\Bigg]G^5\\
&+\Bigg[-\frac{547336457}{4151347200}g(x)\left(\frac{d}{dx}q(x)\right)\frac{d}{dx}g(x)\\
&-\frac{547336457}{1698278400}\left(\frac{d^3}{dx^3}g(x)\right)\frac{d}{dx}q(x)-\frac{547336457}{754790400}\left(\frac{d^4}{dx^4}g(x)\right)q(x)\\
&-\frac{547336457}{24908083200}(g(x))^3q(x)-\frac{9304719769}{14944849920}g(x)q(x)\frac{d^2}{dx^2}g(x)\\
&-\frac{547336457}{1245404160}\left(\frac{d}{dx}g(x)\right)^2q(x)\Bigg]G^4\\
&+\Bigg[-\frac{547336457}{1556755200}(g(x))^2\left(\frac{d}{dx}q(x)\right)\frac{d}{dx}g(x)\\
&-\frac{311434444033}{93405312000}g(x)q(x)\frac{d^4}{dx^4}g(x)\\
&-\frac{547336457}{471744000}\left(\frac{d^5}{dx^5}g(x)\right)\frac{d}{dx}q(x)-\frac{547336457}{113218560}\left(\frac{d^2}{dx^2}g(x)\right)^2q(x)\\
&-\frac{547336457}{18681062400}(g(x))^4q(x)-\frac{547336457}{345945600}g(x)\left(\frac{d}{dx}q(x)\right)\frac{d^3}{dx^3}g(x)\\
&-\frac{547336457}{359251200}(g(x))^2q(x)\frac{d^2}{dx^2}g(x)\\
&-\frac{547336457}{172972800}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}q(x)\right)\frac{d^2}{dx^2}g(x)\\
&-\frac{39955561361}{18681062400}g(x)q(x)\left(\frac{d}{dx}g(x)\right)^2
\end{aligned}$$

$$
\begin{aligned}
&- \frac{69511730039}{9340531200} \left(\frac{d}{dx} g(x)\right) q(x) \frac{d^3}{dx^3} g(x) \\
&- \frac{547336457}{404352000} \left(\frac{d^6}{dx^6} g(x)\right) q(x) \Bigg] G^3 \\
&+ \Bigg[- \frac{547336457}{194594400} (g(x))^2 \left(\frac{d}{dx} q(x)\right) \frac{d^3}{dx^3} g(x) \\
&- \frac{6075982009157}{373621248000} g(x) q(x) \left(\frac{d^2}{dx^2} g(x)\right)^2 \\
&- \frac{547336457}{24908083200} (g(x))^5 q(x) - \frac{12588738511}{849139200} \left(\frac{d^3}{dx^3} g(x)\right)^2 q(x) \\
&- \frac{547336457}{48648600} g(x) \left(\frac{d}{dx} q(x)\right) \left(\frac{d}{dx} g(x)\right) \frac{d^2}{dx^2} g(x) \\
&- \frac{173505656869}{6918912000} g(x) q(x) \left(\frac{d}{dx} g(x)\right) \\
&\frac{d^3}{dx^3} g(x) - \frac{15872757253}{849139200} \left(\frac{d^2}{dx^2} g(x)\right) \left(\frac{d}{dx} q(x)\right) \frac{d^3}{dx^3} g(x) \\
&- \frac{361789398077}{31135104000} \left(\frac{d}{dx} g(x)\right) \left(\frac{d}{dx} q(x)\right) \frac{d^4}{dx^4} g(x) \\
&- \frac{137381450707}{5660928000} \left(\frac{d^2}{dx^2} g(x)\right) q(x) \frac{d^4}{dx^4} g(x) \\
&- \frac{29008832221}{2223936000} \left(\frac{d}{dx} g(x)\right) q(x) \frac{d^5}{dx^5} g(x) \\
&- \frac{147233506933}{37362124800} g(x) \left(\frac{d}{dx} q(x)\right) \frac{d^5}{dx^5} g(x) \\
&- \frac{1648030072027}{373621248000} g(x) q(x) \frac{d^6}{dx^6} g(x) \\
&- \frac{23535467651}{6227020800} (g(x))^2 q(x) \left(\frac{d}{dx} g(x)\right)^2 \\
&- \frac{33387523877}{1556755200} \left(\frac{d}{dx} g(x)\right)^2 q(x) \\
&\frac{d^2}{dx^2} g(x) - \frac{547336457}{1245404160} (g(x))^3 \left(\frac{d}{dx} q(x)\right) \\
&\frac{d}{dx} g(x) - \frac{351937341851}{62270208000} (g(x))^2 q(x) \\
&\frac{d^4}{dx^4} g(x) - \frac{22440794737}{12454041600} (g(x))^3 q(x) \frac{d^2}{dx^2} g(x) \\
&- \frac{547336457}{754790400} \left(\frac{d^8}{dx^8} g(x)\right) q(x) - \frac{547336457}{471744000} \left(\frac{d^7}{dx^7} g(x)\right) \frac{d}{dx} q(x)
\end{aligned}
$$

$$-\frac{547336457}{194594400}\left(\frac{d}{dx}g(x)\right)^3\frac{d}{dx}q(x)\Bigg]G^2$$

$$+\Bigg[-\frac{25724813479}{849139200}\left(\frac{d^3}{dx^3}g(x)\right)\left(\frac{d}{dx}q(x)\right)\frac{d^4}{dx^4}g(x)$$

$$-\frac{69511730039}{5660928000}\left(\frac{d^2}{dx^2}g(x)\right)q(x)\frac{d^6}{dx^6}g(x)$$

$$-\frac{22440794737}{691891200}\left(\frac{d}{dx}g(x)\right)^2q(x)\frac{d^4}{dx^4}g(x)$$

$$-\frac{1454272966249}{46702656000}\left(\frac{d}{dx}g(x)\right)^2\left(\frac{d}{dx}q(x)\right)\frac{d^3}{dx^3}g(x)$$

$$-\frac{59659673813}{2830464000}\left(\frac{d^2}{dx^2}g(x)\right)\left(\frac{d}{dx}q(x)\right)\frac{d^5}{dx^5}g(x)$$

$$-\frac{43239580103}{4447872000}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}q(x)\right)\frac{d^6}{dx^6}g(x)$$

$$-\frac{547336457}{5660928000}\left(\frac{d^{10}}{dx^{10}}g(x)\right)q(x)-\frac{547336457}{1698278400}\left(\frac{d^9}{dx^9}g(x)\right)\frac{d}{dx}q(x)$$

$$-\frac{547336457}{47174400}\left(\frac{d^4}{dx^4}g(x)\right)^2q(x)-\frac{547336457}{111196800}\left(\frac{d}{dx}g(x)\right)^4q(x)$$

$$-\frac{38860888447}{37362124800}(g(x))^4q(x)\frac{d^2}{dx^2}g(x)$$

$$-\frac{20251448909}{3113510400}g(x)\left(\frac{d}{dx}q(x)\right)\left(\frac{d}{dx}g(x)\right)^3$$

$$-\frac{130813413223}{31135104000}(g(x))^3q(x)\frac{d^4}{dx^4}g(x)$$

$$-\frac{411049679207}{93405312000}(g(x))^2\left(\frac{d}{dx}q(x)\right)\frac{d^5}{dx^5}g(x)$$

$$-\frac{58565000899}{23351328000}g(x)\left(\frac{d}{dx}q(x)\right)\frac{d^7}{dx^7}g(x)$$

$$-\frac{9215503926509}{93405312000}\left(\frac{d}{dx}g(x)\right)q(x)\left(\frac{d^3}{dx^3}g(x)\right)\frac{d^2}{dx^2}g(x)$$

$$-\frac{43239580103}{1037836800}g(x)\left(\frac{d}{dx}q(x)\right)\left(\frac{d^2}{dx^2}g(x)\right)\frac{d^3}{dx^3}g(x)$$

$$-\frac{6020701027}{303264000}\left(\frac{d^3}{dx^3}g(x)\right)q(x)\frac{d^5}{dx^5}g(x)$$

$$-\frac{23535467651}{4447872000}\left(\frac{d}{dx}g(x)\right)q(x)\frac{d^7}{dx^7}g(x)$$

$$
\begin{aligned}
&-\frac{115487992427}{5660928000}\left(\frac{d^2}{dx^2}g(x)\right)^3 q(x)\\
&-\frac{20251448909}{9340531200}(g(x))^3\left(\frac{d}{dx}q(x)\right)\frac{d^3}{dx^3}g(x)\\
&-\frac{105635936201}{3335904000}g(x)\,q(x)\left(\frac{d^3}{dx^3}g(x)\right)^2\\
&-\frac{547336457}{114960384}(g(x))^2 q(x)\frac{d^6}{dx^6}g(x)-\frac{547336457}{2075673600}(g(x))^4\left(\frac{d}{dx}q(x)\right)\\
&\frac{d}{dx}g(x)-\frac{547336457}{188697600}(g(x))^3 q(x)\left(\frac{d}{dx}g(x)\right)^2\\
&-\frac{3366666547007}{186810624000}(g(x))^2 q(x)\left(\frac{d^2}{dx^2}g(x)\right)^2\\
&-\frac{20251448909}{13343616000}g(x)\,q(x)\frac{d^8}{dx^8}g(x)-\frac{547336457}{62270208000}(g(x))^6 q(x)\\
&-\frac{1211255579341}{46702656000}g(x)\left(\frac{d}{dx}q(x)\right)\left(\frac{d}{dx}g(x)\right)\frac{d^4}{dx^4}g(x)\\
&-\frac{12588738511}{242611200}g(x)\,q(x)\left(\frac{d^2}{dx^2}g(x)\right)\frac{d^4}{dx^4}g(x)\\
&-\frac{104541263287}{3736212480}g(x)\,q(x)\left(\frac{d}{dx}g(x)\right)\frac{d^5}{dx^5}g(x)\\
&-\frac{401197622981}{8491392000}g(x)\,q(x)\left(\frac{d}{dx}g(x)\right)^2\frac{d^2}{dx^2}g(x)\\
&-\frac{20251448909}{1556755200}(g(x))^2\left(\frac{d}{dx}q(x)\right)\left(\frac{d}{dx}g(x)\right)\frac{d^2}{dx^2}g(x)\\
&-\frac{866433611431}{31135104000}(g(x))^2 q(x)\left(\frac{d}{dx}g(x)\right)\frac{d^3}{dx^3}g(x)\\
&-\frac{1253947822987}{31135104000}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}q(x)\right)\left(\frac{d^2}{dx^2}g(x)\right)^2\Bigg]\,G\\
&-\frac{262174162903}{15567552000}(g(x))^2 q(x)\left(\frac{d^3}{dx^3}g(x)\right)^2\\
&-\frac{148328179847}{12454041600}g(x)\,q(x)\left(\frac{d^4}{dx^4}g(x)\right)^2\\
&-\frac{469067343649}{16982784000}(g(x))^2 q(x)\left(\frac{d^2}{dx^2}g(x)\right)\frac{d^4}{dx^4}g(x)\\
&-\frac{1395160628893}{93405312000}(g(x))^2 q(x)\left(\frac{d}{dx}g(x)\right)\frac{d^5}{dx^5}g(x)
\end{aligned}
$$

$$- \frac{9304719769}{11321856000} \left(\frac{d^2}{dx^2} g(x)\right) q(x) \frac{d^8}{dx^8} g(x)$$
$$- \frac{636552299491}{373621248000} (g(x))^3 q(x) \frac{d^6}{dx^6} g(x)$$
$$- \frac{547336457}{889574400} (g(x))^4 \left(\frac{d}{dx} q(x)\right) \frac{d^3}{dx^3} g(x)$$
$$- \frac{12588738511}{37362124800} g(x) \left(\frac{d}{dx} q(x)\right) \frac{d^9}{dx^9} g(x)$$
$$- \frac{547336457}{8895744000} (g(x))^5 \left(\frac{d}{dx} q(x)\right) \frac{d}{dx} g(x)$$
$$- \frac{1024066511047}{186810624000} g(x) q(x) \left(\frac{d^7}{dx^7} g(x)\right) \frac{d}{dx} g(x)$$
$$- \frac{1575781659703}{124540416000} g(x) q(x) \left(\frac{d^6}{dx^6} g(x)\right) \frac{d^2}{dx^2} g(x)$$
$$- \frac{1905278206817}{93405312000} g(x) q(x) \left(\frac{d^5}{dx^5} g(x)\right) \frac{d^3}{dx^3} g(x)$$
$$- \frac{344274631453}{13343616000} (g(x))^2 q(x) \left(\frac{d}{dx} g(x)\right)^2 \frac{d^2}{dx^2} g(x)$$
$$- \frac{547336457}{111196800} (g(x))^3 \left(\frac{d}{dx} q(x)\right) \left(\frac{d}{dx} g(x)\right) \frac{d^2}{dx^2} g(x)$$
$$- \frac{36671542619}{373621248000} g(x) q(x) \frac{d^{10}}{dx^{10}} g(x) - \frac{272026219129}{26687232000} (g(x))^3 q(x)$$
$$\left(\frac{d}{dx} g(x)\right) \frac{d^3}{dx^3} g(x) - \frac{547336457}{373621248000} \left(\frac{d^{12}}{dx^{12}} g(x)\right) q(x)$$
$$- \frac{547336457}{31135104000} \left(\frac{d^{11}}{dx^{11}} g(x)\right) \frac{d}{dx} q(x)$$
$$- \frac{1067853427607}{31135104000} \left(\frac{d}{dx} g(x)\right)^2 q(x) \left(\frac{d^2}{dx^2} g(x)\right)^2$$
$$- \frac{547336457}{23328000} \left(\frac{d^2}{dx^2} g(x)\right) q(x) \left(\frac{d^3}{dx^3} g(x)\right)^2$$
$$- \frac{250132760849}{6671808000} \left(\frac{d}{dx} g(x)\right) \left(\frac{d}{dx} q(x)\right) \left(\frac{d^3}{dx^3} g(x)\right)^2$$
$$- \frac{6465685566541}{62270208000} g(x) q(x) \left(\frac{d}{dx} g(x)\right) \left(\frac{d^2}{dx^2} g(x)\right) \frac{d^3}{dx^3} g(x)$$
$$- \frac{547336457}{102643200} g(x) q(x) \left(\frac{d}{dx} g(x)\right)^4 - \frac{351937341851}{53374464000} (g(x))^3 q(x)$$

$$\left(\frac{d^2}{dx^2}g(x)\right)^2 - \frac{547336457}{667180800}(g(x))^4 q(x)\left(\frac{d}{dx}g(x)\right)^2$$
$$- \frac{296109023237}{373621248000}(g(x))^2 q(x)\frac{d^8}{dx^8}g(x)$$
$$- \frac{547336457}{148262400}(g(x))^2\left(\frac{d}{dx}q(x)\right)\left(\frac{d}{dx}g(x)\right)^3$$
$$- \frac{12588738511}{9340531200}(g(x))^2\left(\frac{d}{dx}q(x)\right)\frac{d^7}{dx^7}g(x)$$
$$- \frac{16967430167}{494208000}g(x)\left(\frac{d}{dx}q(x)\right)\left(\frac{d}{dx}g(x)\right)^2\frac{d^3}{dx^3}g(x)$$
$$- \frac{547336457}{15966720}g(x)q(x)\left(\frac{d}{dx}g(x)\right)^2\frac{d^4}{dx^4}g(x)$$
$$- \frac{3831355199}{166795200}(g(x))^2\left(\frac{d}{dx}q(x)\right)\left(\frac{d^2}{dx^2}g(x)\right)\frac{d^3}{dx^3}g(x)$$
$$- \frac{191020423493}{13343616000}(g(x))^2\left(\frac{d}{dx}q(x)\right)\left(\frac{d}{dx}g(x)\right)\frac{d^4}{dx^4}g(x)$$
$$- \frac{394629585497}{8895744000}g(x)\left(\frac{d}{dx}q(x)\right)\left(\frac{d}{dx}g(x)\right)\left(\frac{d^2}{dx^2}g(x)\right)^2$$
$$- \frac{278594256613}{26687232000}g(x)\left(\frac{d}{dx}q(x)\right)\left(\frac{d^6}{dx^6}g(x)\right)\frac{d}{dx}g(x)$$
$$- \frac{200872479719}{8895744000}g(x)\left(\frac{d}{dx}q(x)\right)\left(\frac{d^5}{dx^5}g(x)\right)\frac{d^2}{dx^2}g(x)$$
$$- \frac{43239580103}{1334361600}g(x)\left(\frac{d}{dx}q(x)\right)\left(\frac{d^4}{dx^4}g(x)\right)\frac{d^3}{dx^3}g(x)$$
$$- \frac{3831355199}{61776000}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}q(x)\right)\left(\frac{d^4}{dx^4}g(x)\right)\frac{d^2}{dx^2}g(x)$$
$$- \frac{25724813479}{972972000}\left(\frac{d}{dx}g(x)\right)q(x)\left(\frac{d^5}{dx^5}g(x)\right)\frac{d^2}{dx^2}g(x)$$
$$- \frac{449363231197}{12454041600}\left(\frac{d}{dx}g(x)\right)q(x)\left(\frac{d^4}{dx^4}g(x)\right)\frac{d^3}{dx^3}g(x)$$
$$- \frac{61849019641}{53374464000}(g(x))^4 q(x)\frac{d^4}{dx^4}g(x)$$
$$- \frac{2663886536219}{124540416000}g(x)q(x)\left(\frac{d^2}{dx^2}g(x)\right)^3$$
$$- \frac{12588738511}{53374464000}(g(x))^5 q(x)\frac{d^2}{dx^2}g(x)$$

$$
\begin{aligned}
&-\frac{43239580103}{26687232000}\,(g(x))^3\left(\frac{d}{dx}q(x)\right)\frac{d^5}{dx^5}g(x)\\
&-\frac{12588738511}{8491392000}\left(\frac{d^3}{dx^3}g(x)\right)q(x)\,\frac{d^7}{dx^7}g(x)\\
&-\frac{6020701027}{1415232000}\left(\frac{d^2}{dx^2}g(x)\right)\left(\frac{d}{dx}q(x)\right)\frac{d^7}{dx^7}g(x)\\
&-\frac{33387523877}{20756736000}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}q(x)\right)\frac{d^8}{dx^8}g(x)\\
&-\frac{9304719769}{1213056000}\left(\frac{d^3}{dx^3}g(x)\right)\left(\frac{d}{dx}q(x)\right)\frac{d^6}{dx^6}g(x)\\
&-\frac{12588738511}{741312000}\left(\frac{d}{dx}g(x)\right)^2\left(\frac{d}{dx}q(x)\right)\frac{d^5}{dx^5}g(x)\\
&-\frac{15872757253}{46702656000}\left(\frac{d}{dx}g(x)\right)q(x)\,\frac{d^9}{dx^9}g(x)\\
&-\frac{23535467651}{11321856000}\left(\frac{d^4}{dx^4}g(x)\right)q(x)\,\frac{d^6}{dx^6}g(x)\\
&-\frac{547336457}{25272000}\left(\frac{d}{dx}g(x)\right)^3\left(\frac{d}{dx}q(x)\right)\frac{d^2}{dx^2}g(x)\\
&-\frac{219481919257}{31135104000}\left(\frac{d}{dx}g(x)\right)^2 q(x)\,\frac{d^6}{dx^6}g(x)\\
&-\frac{547336457}{471744000}\left(\frac{d^5}{dx^5}g(x)\right)^2 q(x)-\frac{547336457}{373621248000}\,(g(x))^7\,q(x)\\
&-\frac{547336457}{28740096}\left(\frac{d}{dx}g(x)\right)^3 q(x)\,\frac{d^3}{dx^3}g(x)\\
&-\frac{230428648397}{11321856000}\left(\frac{d^2}{dx^2}g(x)\right)^2 q(x)\,\frac{d^4}{dx^4}g(x)\\
&-\frac{3831355199}{80870400}\left(\frac{d^2}{dx^2}g(x)\right)^2\left(\frac{d}{dx}q(x)\right)\frac{d^3}{dx^3}g(x)\\
&-\frac{547336457}{53913600}\left(\frac{d^4}{dx^4}g(x)\right)\left(\frac{d}{dx}q(x)\right)\frac{d^5}{dx^5}g(x)\Bigg]
\end{aligned}
\tag{45}
$$

The new proposed method of the family developed in Sect. 3.2 (mentioned as *PFDF*)

$$
\begin{aligned}
&\mathrm{LTE}_{\mathrm{PFDF}}\\
&=h^{14}\Bigg[\Bigg[-\frac{547336457}{17791488000}\left(\frac{d^2}{dx^2}g(x)\right)q(x)-\frac{547336457}{373621248000}\,(g(x))^2\,q(x)
\end{aligned}
$$

$$-\frac{547336457}{186810624000}\left(\frac{d}{dx}g(x)\right)\frac{d}{dx}q(x)\Bigg]G^5+\Bigg[-\frac{547336457}{12454041600}g(x)\left(\frac{d}{dx}q(x)\right)\frac{d}{dx}g(x)$$

$$-\frac{10399392683}{24908083200}\left(\frac{d^4}{dx^4}g(x)\right)q(x)-\frac{547336457}{74724249600}(g(x))^3q(x)$$

$$-\frac{547336457}{2668723200}\left(\frac{d}{dx}g(x)\right)^2q(x)-\frac{547336457}{1916006400}g(x)q(x)\frac{d^2}{dx^2}g(x)$$

$$-\frac{547336457}{3736212480}\left(\frac{d^3}{dx^3}g(x)\right)\frac{d}{dx}q(x)\Bigg]G^4+\Bigg[-\frac{16967430167}{18681062400}(g(x))^2q(x)$$

$$\frac{d^2}{dx^2}g(x)-\frac{547336457}{583783200}g(x)\left(\frac{d}{dx}q(x)\right)\frac{d^3}{dx^3}g(x)$$

$$-\frac{547336457}{424569600}g(x)q(x)\left(\frac{d}{dx}g(x)\right)^2-\frac{547336457}{239500800}g(x)q(x)\frac{d^4}{dx^4}g(x)$$

$$-\frac{547336457}{3113510400}(g(x))^2\left(\frac{d}{dx}q(x)\right)\frac{d}{dx}g(x)-\frac{547336457}{161740800}\left(\frac{d^2}{dx^2}g(x)\right)^2q(x)$$

$$-\frac{547336457}{37362124800}(g(x))^4q(x)-\frac{547336457}{691891200}\left(\frac{d^5}{dx^5}g(x)\right)\frac{d}{dx}q(x)$$

$$-\frac{547336457}{291891600}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}q(x)\right)\frac{d^2}{dx^2}g(x)$$

$$-\frac{32292850963}{6227020800}\left(\frac{d}{dx}g(x)\right)q(x)\frac{d^3}{dx^3}g(x)-\frac{9304719769}{8895744000}\left(\frac{d^6}{dx^6}g(x)\right)q(x)\Bigg]G^3$$

$$+\Bigg[-\frac{547336457}{41932800}g(x)q(x)\left(\frac{d^2}{dx^2}g(x)\right)^2-\frac{547336457}{415134720}(g(x))^3q(x)\frac{d^2}{dx^2}g(x)$$

$$-\frac{547336457}{266872320}(g(x))^2\left(\frac{d}{dx}q(x)\right)\frac{d^3}{dx^3}g(x)-\frac{32292850963}{10378368000}g(x)\left(\frac{d}{dx}q(x)\right)\frac{d^5}{dx^5}g(x)$$

$$-\frac{10399392683}{3736212480}(g(x))^2q(x)\left(\frac{d}{dx}g(x)\right)^2-\frac{104541263287}{23351328000}(g(x))^2q(x)$$

$$\frac{d^4}{dx^4}g(x)-\frac{547336457}{1868106240}(g(x))^3\left(\frac{d}{dx}q(x)\right)\frac{d}{dx}g(x)-\frac{33387523877}{8895744000}g(x)q(x)\frac{d^6}{dx^6}g(x)$$

$$-\frac{12588738511}{849139200}\left(\frac{d^2}{dx^2}g(x)\right)\left(\frac{d}{dx}q(x)\right)\frac{d^3}{dx^3}g(x)$$

$$-\frac{3831355199}{416988000}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}q(x)\right)\frac{d^4}{dx^4}g(x)-\frac{547336457}{42456960}\left(\frac{d^3}{dx^3}g(x)\right)^2q(x)$$

$$-\frac{547336457}{37362124800}(g(x))^5q(x)-\frac{547336457}{555984000}\left(\frac{d^7}{dx^7}g(x)\right)\frac{d}{dx}q(x)$$

$$-\frac{547336457}{830269440}\left(\frac{d^8}{dx^8}g(x)\right)q(x)-\frac{547336457}{266872320}\left(\frac{d}{dx}g(x)\right)^3\frac{d}{dx}q(x)$$

$$- \frac{327854537743}{15567552000} \left(\frac{d^2}{dx^2} g(x)\right) q(x) \frac{d^4}{dx^4} g(x)$$
$$- \frac{1050338660983}{93405312000} \left(\frac{d}{dx} g(x)\right) q(x) \frac{d^5}{dx^5} g(x)$$
$$- \frac{9304719769}{533744640} \left(\frac{d}{dx} g(x)\right)^2 q(x) \frac{d^2}{dx^2} g(x)$$
$$- \frac{547336457}{66718080} g(x) \left(\frac{d}{dx} q(x)\right) \left(\frac{d}{dx} g(x)\right)$$
$$\left.\frac{d^2}{dx^2} g(x) - \frac{74985094609}{3736212480} g(x) q(x) \left(\frac{d}{dx} g(x)\right) \frac{d^3}{dx^3} g(x)\right] G^2$$
$$+ \left[- \frac{361789398077}{15567552000} g(x) \left(\frac{d}{dx} q(x)\right) \left(\frac{d}{dx} g(x)\right) \frac{d^4}{dx^4} g(x)\right.$$
$$- \frac{15872757253}{424569600} g(x) \left(\frac{d}{dx} q(x)\right) \left(\frac{d^2}{dx^2} g(x)\right) \frac{d^3}{dx^3} g(x)$$
$$- \frac{137381450707}{2830464000} g(x) q(x) \left(\frac{d^2}{dx^2} g(x)\right) \frac{d^4}{dx^4} g(x)$$
$$- \frac{29008832221}{1111968000} g(x) q(x) \left(\frac{d}{dx} g(x)\right) \frac{d^5}{dx^5} g(x)$$
$$- \frac{33387523877}{778377600} g(x) q(x) \left(\frac{d}{dx} g(x)\right)^2 \frac{d^2}{dx^2} g(x)$$
$$- \frac{547336457}{48648600} (g(x))^2 \left(\frac{d}{dx} q(x)\right) \left(\frac{d}{dx} g(x)\right) \frac{d^2}{dx^2} g(x)$$
$$- \frac{173505656869}{6918912000} (g(x))^2 q(x) \left(\frac{d}{dx} g(x)\right) \frac{d^3}{dx^3} g(x)$$
$$- \frac{547336457}{74724249600} (g(x))^6 q(x) - \frac{140665469449}{12454041600} \left(\frac{d^4}{dx^4} g(x)\right)^2 q(x)$$
$$- \frac{17464959006413}{186810624000} \left(\frac{d}{dx} g(x)\right) q(x) \left(\frac{d^3}{dx^3} g(x)\right) \frac{d^2}{dx^2} g(x)$$
$$- \frac{547336457}{5748019200} \left(\frac{d^{10}}{dx^{10}} g(x)\right) q(x) - \frac{547336457}{291891600} (g(x))^3 \left(\frac{d}{dx} q(x)\right) \frac{d^3}{dx^3} g(x)$$
$$- \frac{547336457}{377395200} g(x) q(x) \frac{d^8}{dx^8} g(x) - \frac{147233506933}{37362124800} (g(x))^2 \left(\frac{d}{dx} q(x)\right) \frac{d^5}{dx^5} g(x)$$
$$- \frac{547336457}{97297200} g(x) \left(\frac{d}{dx} q(x)\right) \left(\frac{d}{dx} g(x)\right)^3 - \frac{547336457}{2490808320} (g(x))^4 \left(\frac{d}{dx} q(x)\right)$$
$$\frac{d}{dx} g(x) - \frac{351937341851}{93405312000} (g(x))^3 q(x) \frac{d^4}{dx^4} g(x) - \frac{1648030072027}{373621248000} (g(x))^2 q(x)$$
$$\frac{d^6}{dx^6} g(x) - \frac{2610247563433}{93405312000} \left(\frac{d}{dx} g(x)\right)^2 \left(\frac{d}{dx} q(x)\right) \frac{d^3}{dx^3} g(x)$$

$$- \frac{2253384193469}{62270208000} \left(\frac{d}{dx} g(x)\right)\left(\frac{d}{dx} q(x)\right)\left(\frac{d^2}{dx^2} g(x)\right)^2$$

$$- \frac{15872757253}{518918400} \left(\frac{d}{dx} g(x)\right)^2 q(x) \frac{d^4}{dx^4} g(x) - \frac{6075982009157}{373621248000} (g(x))^2 q(x)$$

$$\left(\frac{d^2}{dx^2} g(x)\right)^2 - \frac{12588738511}{424569600} g(x) q(x)\left(\frac{d^3}{dx^3} g(x)\right)^2$$

$$- \frac{22440794737}{24908083200} (g(x))^4 q(x) \frac{d^2}{dx^2} g(x) - \frac{547336457}{235872000} g(x)\left(\frac{d}{dx} q(x)\right) \frac{d^7}{dx^7} g(x)$$

$$- \frac{23535467651}{9340531200} (g(x))^3 q(x)\left(\frac{d}{dx} g(x)\right)^2 - \frac{547336457}{121305600} \left(\frac{d}{dx} g(x)\right)^4 q(x)$$

$$- \frac{952912771637}{186810624000} \left(\frac{d}{dx} g(x)\right) q(x) \frac{d^7}{dx^7} g(x) - \frac{360694725163}{18681062400} \left(\frac{d^3}{dx^3} g(x)\right) q(x)$$

$$\frac{d^5}{dx^5} g(x) - \frac{240280704623}{26687232000} \left(\frac{d}{dx} g(x)\right)\left(\frac{d}{dx} q(x)\right) \frac{d^6}{dx^6} g(x)$$

$$- \frac{20251448909}{718502400} \left(\frac{d^3}{dx^3} g(x)\right)\left(\frac{d}{dx} q(x)\right) \frac{d^4}{dx^4} g(x)$$

$$- \frac{23535467651}{1976832000} \left(\frac{d^2}{dx^2} g(x)\right) q(x) \frac{d^6}{dx^6} g(x)$$

$$- \frac{1218918289739}{62270208000} \left(\frac{d^2}{dx^2} g(x)\right)\left(\frac{d}{dx} q(x)\right) \frac{d^5}{dx^5} g(x)$$

$$- \frac{345369304367}{17791488000} \left(\frac{d^2}{dx^2} g(x)\right)^3 q(x) - \frac{547336457}{1779148800} \left(\frac{d^9}{dx^9} g(x)\right) \frac{d}{dx} q(x) \Bigg] G$$

$$- \frac{43239580103}{26687232000} (g(x))^3 \left(\frac{d}{dx} q(x)\right) \frac{d^5}{dx^5} g(x)$$

$$- \frac{148328179847}{12454041600} g(x) q(x)\left(\frac{d^4}{dx^4} g(x)\right)^2$$

$$- \frac{25724813479}{972972000} \left(\frac{d}{dx} g(x)\right) q(x)\left(\frac{d^5}{dx^5} g(x)\right) \frac{d^2}{dx^2} g(x)$$

$$- \frac{250132760849}{6671808000} \left(\frac{d}{dx} g(x)\right)\left(\frac{d}{dx} q(x)\right)\left(\frac{d^3}{dx^3} g(x)\right)^2$$

$$- \frac{547336457}{23328000} \left(\frac{d^2}{dx^2} g(x)\right) q(x)\left(\frac{d^3}{dx^3} g(x)\right)^2$$

$$- \frac{12588738511}{741312000} \left(\frac{d}{dx} g(x)\right)^2 \left(\frac{d}{dx} q(x)\right) \frac{d^5}{dx^5} g(x)$$

$$- \frac{547336457}{25272000} \left(\frac{d}{dx} g(x)\right)^3 \left(\frac{d}{dx} q(x)\right) \frac{d^2}{dx^2} g(x)$$

$$- \frac{547336457}{53913600}\left(\frac{d^4}{dx^4}g(x)\right)\left(\frac{d}{dx}q(x)\right)\frac{d^5}{dx^5}g(x)$$
$$- \frac{547336457}{28740096}\left(\frac{d}{dx}g(x)\right)^3 q(x)\frac{d^3}{dx^3}g(x)$$
$$- \frac{547336457}{148262400}(g(x))^2\left(\frac{d}{dx}q(x)\right)\left(\frac{d}{dx}g(x)\right)^3$$
$$- \frac{449363231197}{12454041600}\left(\frac{d}{dx}g(x)\right)q(x)\left(\frac{d^4}{dx^4}g(x)\right)\frac{d^3}{dx^3}g(x)$$
$$- \frac{547336457}{111196800}(g(x))^3\left(\frac{d}{dx}q(x)\right)\left(\frac{d}{dx}g(x)\right)\frac{d^2}{dx^2}g(x)$$
$$- \frac{272026219129}{26687232000}(g(x))^3 q(x)\left(\frac{d}{dx}g(x)\right)\frac{d^3}{dx^3}g(x)$$
$$- \frac{262174162903}{15567552000}(g(x))^2 q(x)\left(\frac{d^3}{dx^3}g(x)\right)^2$$
$$- \frac{3831355199}{80870400}\left(\frac{d^2}{dx^2}g(x)\right)^2\left(\frac{d}{dx}q(x)\right)\frac{d^3}{dx^3}g(x)$$
$$- \frac{15872757253}{46702656000}\left(\frac{d}{dx}g(x)\right)q(x)\frac{d^9}{dx^9}g(x)$$
$$- \frac{23535467651}{11321856000}\left(\frac{d^4}{dx^4}g(x)\right)q(x)\frac{d^6}{dx^6}g(x)$$
$$- \frac{2663886536219}{124540416000}g(x)q(x)\left(\frac{d^2}{dx^2}g(x)\right)^3$$
$$- \frac{547336457}{667180800}(g(x))^4 q(x)\left(\frac{d}{dx}g(x)\right)^2$$
$$- \frac{296109023237}{373621248000}(g(x))^2 q(x)\frac{d^8}{dx^8}g(x)$$
$$- \frac{1024066511047}{186810624000}g(x)q(x)\left(\frac{d^7}{dx^7}g(x)\right)\frac{d}{dx}g(x)$$
$$- \frac{1395160628893}{93405312000}(g(x))^2 q(x)\left(\frac{d}{dx}g(x)\right)\frac{d^5}{dx^5}g(x)$$
$$- \frac{547336457}{373621248000}\left(\frac{d^{12}}{dx^{12}}g(x)\right)q(x)$$
$$- \frac{43239580103}{1334361600}g(x)\left(\frac{d}{dx}q(x)\right)\left(\frac{d^4}{dx^4}g(x)\right)\frac{d^3}{dx^3}g(x)$$
$$- \frac{9304719769}{1213056000}\left(\frac{d^3}{dx^3}g(x)\right)\left(\frac{d}{dx}q(x)\right)\frac{d^6}{dx^6}g(x)$$

$$
\begin{aligned}
&-\frac{9304719769}{11321856000}\left(\frac{d^2}{dx^2}g(x)\right)q(x)\frac{d^8}{dx^8}g(x)\\
&-\frac{61849019641}{53374464000}(g(x))^4\,q(x)\frac{d^4}{dx^4}g(x)\\
&-\frac{36671542619}{373621248000}g(x)\,q(x)\frac{d^{10}}{dx^{10}}g(x)\\
&-\frac{12588738511}{9340531200}(g(x))^2\left(\frac{d}{dx}q(x)\right)\frac{d^7}{dx^7}g(x)\\
&-\frac{351937341851}{53374464000}(g(x))^3\,q(x)\left(\frac{d^2}{dx^2}g(x)\right)^2\\
&-\frac{12588738511}{37362124800}g(x)\left(\frac{d}{dx}q(x)\right)\frac{d^9}{dx^9}g(x)\\
&-\frac{6465685566541}{62270208000}g(x)\,q(x)\left(\frac{d}{dx}g(x)\right)\left(\frac{d^2}{dx^2}g(x)\right)\frac{d^3}{dx^3}g(x)\\
&-\frac{3831355199}{166795200}(g(x))^2\left(\frac{d}{dx}q(x)\right)\left(\frac{d^2}{dx^2}g(x)\right)\frac{d^3}{dx^3}g(x)\\
&-\frac{12588738511}{8491392000}\left(\frac{d^3}{dx^3}g(x)\right)q(x)\frac{d^7}{dx^7}g(x)\\
&-\frac{6020701027}{1415232000}\left(\frac{d^2}{dx^2}g(x)\right)\left(\frac{d}{dx}q(x)\right)\frac{d^7}{dx^7}g(x)\\
&-\frac{547336457}{15966720}g(x)\,q(x)\left(\frac{d}{dx}g(x)\right)^2\frac{d^4}{dx^4}g(x)\\
&-\frac{547336457}{31135104000}\left(\frac{d^{11}}{dx^{11}}g(x)\right)\frac{d}{dx}q(x)\\
&-\frac{16967430167}{494208000}g(x)\left(\frac{d}{dx}q(x)\right)\left(\frac{d}{dx}g(x)\right)^2\frac{d^3}{dx^3}g(x)\\
&-\frac{191020423493}{13343616000}(g(x))^2\left(\frac{d}{dx}q(x)\right)\left(\frac{d}{dx}g(x)\right)\frac{d^4}{dx^4}g(x)\\
&-\frac{33387523877}{20756736000}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}q(x)\right)\frac{d^8}{dx^8}g(x)\\
&-\frac{230428648397}{11321856000}\left(\frac{d^2}{dx^2}g(x)\right)^2q(x)\frac{d^4}{dx^4}g(x)\\
&-\frac{394629585497}{8895744000}g(x)\left(\frac{d}{dx}q(x)\right)\left(\frac{d}{dx}g(x)\right)\left(\frac{d^2}{dx^2}g(x)\right)^2\\
&-\frac{344274631453}{13343616000}(g(x))^2\,q(x)\left(\frac{d}{dx}g(x)\right)^2\frac{d^2}{dx^2}g(x)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{3831355199}{61776000}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}q(x)\right)\left(\frac{d^4}{dx^4}g(x)\right)\frac{d^2}{dx^2}g(x)\\
&-\frac{219481919257}{31135104000}\left(\frac{d}{dx}g(x)\right)^2 q(x)\frac{d^6}{dx^6}g(x)\\
&-\frac{1067853427607}{31135104000}\left(\frac{d}{dx}g(x)\right)^2 q(x)\left(\frac{d^2}{dx^2}g(x)\right)^2\\
&-\frac{1575781659703}{124540416000}g(x)\,q(x)\left(\frac{d^6}{dx^6}g(x)\right)\frac{d^2}{dx^2}g(x)\\
&-\frac{200872479719}{8895744000}g(x)\left(\frac{d}{dx}q(x)\right)\left(\frac{d^5}{dx^5}g(x)\right)\frac{d^2}{dx^2}g(x)\\
&-\frac{1905278206817}{93405312000}g(x)\,q(x)\left(\frac{d^5}{dx^5}g(x)\right)\frac{d^3}{dx^3}g(x)\\
&-\frac{547336457}{102643200}g(x)\,q(x)\left(\frac{d}{dx}g(x)\right)^4-\frac{547336457}{373621248000}(g(x))^7 q(x)\\
&-\frac{547336457}{889574400}(g(x))^4\left(\frac{d}{dx}q(x)\right)\frac{d^3}{dx^3}g(x)\\
&-\frac{547336457}{8895744000}(g(x))^5\left(\frac{d}{dx}q(x)\right)\frac{d}{dx}g(x)-\frac{636552299491}{373621248000}(g(x))^3 q(x)\\
&\frac{d^6}{dx^6}g(x)-\frac{12588738511}{53374464000}(g(x))^5 q(x)\frac{d^2}{dx^2}g(x)\\
&-\frac{278594256613}{26687232000}g(x)\left(\frac{d}{dx}q(x)\right)\left(\frac{d^6}{dx^6}g(x)\right)\frac{d}{dx}g(x)\\
&-\frac{547336457}{471744000}\left(\frac{d^5}{dx^5}g(x)\right)^2 q(x)-\frac{469067343649}{16982784000}(g(x))^2 q(x)\\
&\left.\left(\frac{d^2}{dx^2}g(x)\right)\frac{d^4}{dx^4}g(x)\right] \qquad (46)
\end{aligned}
$$

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